PRACTICAL MATHEMATICS

ISSUE No. 15

A QUARTERLY

WITH APPLICATIONS IN THE FIELD OF BUSINESS

MATHEMATICS OF LIFE INSURANCE

Life Annuities and Life Insurance Net Premiums

Annuities Certain - Life Annuities
Varying Insurances and Annuities

Life Insurance Reserves and Non-Forfeiture Benefits

By C. J. NESBITT, Ph.D.

Net Level Premium Reserves
Modified Reserves - Dividends

—ALSO—

Insurance Tables and Formulas
Glossary of Insurance Terms
Self-Tests and Mathematics Problems

CARL H. FISCHER, Ph.D.
University of Michigan

50¢

EDITOR: REGINALD STEVENS KIMBALL ED.D.

Issue No. 15 PRACTICAL MATHEMATICS Volume No. 3

ISSUE 15

Practical Mathematics

VOLUME 3

REGINALD STEVENS KIMBALL, Editor

TABLE OF CONTENTS

PRACTICAL MATHEMATICS is published quarterly by the National Educational Alliance, Inc. Office of publication at Washington and South Avenues, Dunellen, N. J. Executive and editorial offices, 37 W. 47th St., New York 19, N. Y. John J. Crawley, President; A. R. Mahony, Vice President and Business Manager; Frank P. Crawley, Treasurer. Issue No. 15, November 20, 1943. Entered as second-class matter April 10, 1943, at the Post Office at Dunellen, N. J., under the Act of March 3, 1879. Printed in the U.S.A. Price in the U.S.A., 50c a copy; annual subscription at the rate of 50c a copy.

CHATS WITH THE EDITOR

SOONER or later, every man and woman is visited by an insurance agent who solicits him to take out life insurance with the company which he represents. Most of us have deemed it prudent to carry some amount of insurance, large or small, depending upon our income and the other demands upon our pocketbooks, particularly if we have a family dependent upon us.

The average man is almost entirely dependent upon the advice of the insurance agent in selecting the form of policy. Those of us who are fortunate enough to secure the services of a conscientious and well-informed agent possess policies which are not unduly burdensome so far as premium payments are concerned and which afford a degree of protection for the beneficiaries whom we have named.

To most of us, however, the whole subject of life insurance is a mystery. We know, dimly, that the theory behind the system is predicated on the fact that, of a large number of given cases, a certain number may be expected to outlive others. By striking an "average", payments are determined whereby the amounts collected as premiums will be sufficient to meet any obligations for policy payments which a company may incur from time to time.

In modern life insurance practice, the determination of these rates is not a subject for guess-work. Mathematicians are employed to make accurate forecasts of the situation. Such men are called actuaries. The work of these actuaries is highly important, for on their findings and recommendations depend the financial security of the companies they serve and the policyholders who insure therein.

A careful reading of the articles in the present issue will be of assistance to almost all of our readers. Those of us who do not expect to become life insurance actuaries may not want to burden ourselves with the formal study of this issue, but we shall all want to profit from an understanding of how the various types of policies and premiums are computed. For those of our number who expect to make a career in life insurance, the value of the present issue is incalculable. Here, in brief compass, two of the outstanding professors of actuarial mathematics present a simple but comprehensive statement of the principles and mathematical computations which underlie the subject. With the help of their illustrative examples and diagrams, the student is enabled to gain an introduction to the subject which will pave the way for more prolonged study at some future time.

Before undertaking the study of the present issue, readers of PRACTICAL MATHEMATICS are advised to review the issues on fundamentals of arithmetic and algebra, paying especial attention to the sections on logarithms (pages 88 to 100) and the use of the slide rule (pages 101 to 114). Many of the computations which are involved in the insurance articles presented in this issue may be solved much more readily with the assistance of these devices for shortening

the labor of computing multiplications and divisions. In the offices of insurance companies, much of this computation is done by machines which, unfortunately, are not generally available to the average reader.

As in most of our issues, we again present some simplified tables which will be of assistance to the reader. Again, we remind you that there are available longer and more complete tables to which you will want to turn when you have need of greater accuracy than these five- or six-place tables afford. For all practical purposes, however, you will find that the use of the tables in this issue gives all the practice needed and that the results obtained with them are reasonably correct. The solutions printed in the illustrative examples and on the answers page are based upon the tables which are given here; in a few instances, as noted, use of larger tables might give a difference in the last significant figure. Since this difference, in the examples chosen, would amount to not more than a cent or two in the final answer, it will be seen that these tables are reasonably accurate.

At first glance, the reader may be somewhat appalled by the large number of formulas which appear in the articles on life insurance. It is not necessary to memorize all of them, fortunately, for, as Dr. Fischer points out on page 913, many of these are presented merely for the purpose of showing how one step leads to another. It is good practice to read carefully enough to be able to see how each formula is developed out of those which precede, but to memorize only those which the authors signal out for special attention.

In approaching the field of life insurance, the reader will encounter a few new symbols which have not been employed in the theoretical mathematics in our earlier issues. In some of the subscripts, an angle, or half-square ($\urcorner$), appears. This resembles one of the forms for factorial numbers (see Issue Number Four, page 208), but here its use is altogether different. In the insurance formulas, the presence of an angle over a subscript makes the letter or numeral signify a term, while its absence makes it signify an age. Care should be taken, then, to write the half-square whenever it should appear. It is an important part of the formula.

The use of the small italic *a* signifies that annuity payments are made at the ends of the payment intervals, while the use of the small Roman a signifies that payments are made at the beginnings of the intervals. Actuaries distinguish between these two forms by calling them "round a" and "curly a". The reader should note that some insurance texts have other symbols for the annuity immediate.

Two other letters which may appear slightly strange to the reader are the "open N" ($\mathbb{N}$) and the "open S" ($\mathbb{S}$). Their significance in the field of insurance is adequately treated by the authors on the occasion of their first appearance in the text (pages 909 and 933); hence, no detailed explanation is necessary here. The "bar S" ($\$$) is another new symbol, not to be confused with the dollar sign ($).

In connection with the final section of Dr. Nesbitt's article, dealing with dividends (pages 978 to 980), the reader will want to bear in mind that there are two kinds of life insurance companies: mutual and stock. A mutual company is owned by its policyholders and is, in that sense, a coöperative enterprise. A mutual company issues participating policies which entitle the policyholders to share in the profits being distributed in the form of annual dividends. A stock company is owned by its stockholders, and usually issues non-

participating policies, although there are stock companies which issue both participating and non-participating insurance. As the name suggests, under a non-participating policy the policyholder does not share in the profits of the insurance operations. Instead, if the stock company issues only non-participating insurance, these profits are the property of the stockholders.

There is a standing controversy concerning the merits of the two kinds of insurance. As it works out in practice, the non-participating premium rates, through the force of competition among companies, are computed as closely as is reasonable to the most probable interest, mortality, and expense factors. The non-participating premium rates are then as low as reasonable caution permits, and the stockholders bear the risk that the premiums will be insufficient to meet the costs of the insurance operations. The participating premium rates are, on the other hand, computed quite conservatively, and with no special attempt to reach the minimum rates. The usually substantial profits arising from the excess of premiums over the cost of insurance operations are distributed as equitably as possible by means of annual dividends. The extra margins in the participating premiums are available in case costs are higher than anticipated; in that event, all that happens is that dividends are reduced. For a given insurance benefit, the participating premium will be higher than the non-participating premium, but the mutual company's representative will be quick to point out that the net cost—that is, annual premium less annual dividend—may compare quite favorably with the non-participating premium.

Upon their return to civilian life, many of our service men will probably, if we may judge from the experiences following World War I, convert their insurance into one form or another of those offered by the federal government. Many others, judging from the same experience, will terminate their government insurance but, mindful of the lessons on insurance which they have received while in the service, will sooner or later take out policies in some of the old-line companies. The very fact that the United States Government stresses the value of insurance and presents the facts to the members of the armed forces so clearly will give an impetus to the insurance business which will make it an attractive field for many who are in search of a livelihood.

While this issue deals almost entirely with life insurance, the reader should realize that there are many other forms of insurance, each of which has its own formulas and theory based on compilations of large numbers of cases. Since there is an element of unpredictability in most other forms of insurance, a certain amount of unreliability creeps into the figuring of risks and a greater degree of "guess-work" enters into the calculations.

In order that the reader of this issue may see how life insurance fits into the general insurance field, it may be well to review briefly the customary classifications of insurance. The insurance field as a whole has two subdivisions: social and voluntary. Under social insurance, usually operated by the federal government or by the various states, such matters as old age and survivors' insurance, employment security, and industrial accident are usually covered.

Voluntary insurance is offered by commercial companies, by coöperative associations, and by the government. It has two main classifications, personal and property. Personal insur-

ance covers, in addition to life insurance, other forms, such as accident and health insurance. Property insurance covers the subdivisions of marine insurance, fire insurance, casualty insurance, and sureties.

It should be noted that the insurance of industrial accidents, while regarded as a form of social insurance, is largely written by commercial carriers.

R.S.K.

ABOUT OUR AUTHORS

CARL H. FISCHER, the author of the first article in this issue, pursued his undergraduate studies and secured the degree of Bachelor of Science at Washington University, St. Louis, Missouri. He later specialized in statistics and actuarial theory, studying under Professor Henry L. Rietz at the University of Iowa, the institution at which he was awarded the degrees of Master of Science and Doctor of Philosophy. He has taught at Beloit College, the University of Iowa, the University of Minnesota, and Wayne University. He is at present an Assistant Professor of Mathematics at the University of Michigan, where, in company with Dr. Nesbitt, the other author whose work is presented in this issue, he teaches actuarial work.

Dr. Fischer has engaged in actuarial research for the Northwestern National Life Insurance Company at its home office in Minneapolis and has been actuarial consultant in connection with several surveys of large pension funds for teachers.

He is a member of the Fraternal Actuarial Association, the Institute of Mathematical Statistics, the American Statistical Association, the American Mathematical Society, and the Mathematical Association of America. He has also been honored by election to Sigma Xi, the national honorary scientific society.

CECIL JAMES NESBITT was born at Fort William, Ontario, but moved at an early age to Edmonton, Alberta, where he received most of his elementary and secondary schooling. Advised by his high-school principal to pursue actuarial training, he enrolled at the University of Toronto, taking courses in actuarial science under the tutelage of Professors M. A. MacKenzie and N. E. Sheppard. Upon his graduation in 1934, he received the medal and prize of the British Association for the Advancement of Science which is annually awarded to a student graduating in mathematics. For the next three years, while under appointment as a teaching fellow in the department of mathematics, he completed the requirements for the degree of Doctor of Philosophy, writing a thesis in algebra under the direction of Professor R. Brauer.

In 1937, Dr. Nesbitt was appointed a member of the Institute for Advanced Study, located at Princeton, New Jersey, where, in close contact with Professor Tadasi Nakayama, he engaged in post-doctoral study in the field of mathematics.

He has served at the University of Michigan since 1938, being appointed an Assistant Professor in 1941. Although his teaching has been mainly in the field of actuarial mathematics, most of Dr. Nesbitt's researches and writings have been in the field of algebra.

Dr. Nesbitt is a member of the American Mathematical Society, the Institute of Mathematical Statistics, the Michigan Actuarial Society, and the American Institute of Actuaries. He is also an Associate of the Actuarial Society of America.

· LIFE ANNUITIES AND LIFE INSURANCE NET PREMIUMS ·

By Carl H. Fischer, Ph.D.

WHILE much of the computation necessary for insurance calculations can be worked out laboriously by means of simple arithmetic, a knowledge of some of the formulas from algebra and higher branches of mathematics will frequently effect a saving of time and will often add to one's comprehension of the underlying principles. Before proceeding to memorize any of the material in this article, the reader is advised to read it straight through, later returning to the more careful study of the various sections.—*Editor.*

COMPOUND INTEREST AND ANNUITIES CERTAIN

In life insurance contracts, the initial payment is always made by the insured person to the insurer; in fact, in the vast majority of instances, the insured makes many payments to the company before he or his beneficiary receives any return. Thus, the insurance company is commonly in the position of being a custodian of sums of money which will eventually be repaid. In practice, this money is always invested at interest (usually in high-grade bonds or similar secure investments), and hence it increases with time. The mathematical theory of life insurance recognizes this productive use of money by the insurer. In all actuarial computations, it is assumed that money is constantly invested at compound interest.

Compound interest

In transactions involving compound interest, it is mutually agreed by debtor and creditor that interest shall be computed at certain stated intervals but that, instead of being paid at those times, it shall be added to and become a part of the invested principal. From this, it follows that each succeeding interest computation will be based upon a larger principal than those that precede.

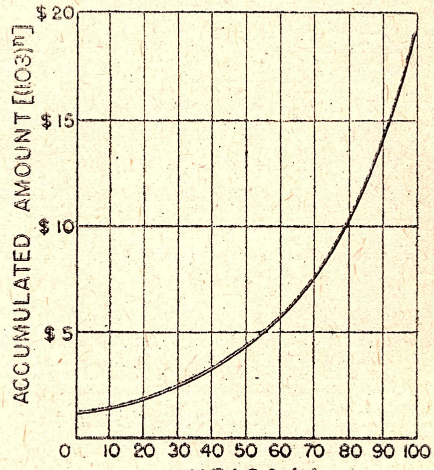
To illustrate, if it is agreed that interest shall be at 5%, compounded annually, on a loan of $100 for 3 years, the first year's interest will amount to $5. This sum now becomes part of the invested principal. The interest for the second year is 5% of the new principal of $105; this amounts to $5.25,

making the principal equal to \$110.25 at the beginning of the third year. The third year's interest is \$5.51, which, added to the \$110.25, totals \$115.76 as the amount to be paid to the creditor at the end of the third year to discharge the debt.

Because of the fact that life insurance computations are based upon an interest rate which is compounded annually, we shall restrict the following discussion of compound interest to this case.

If the annual rate of interest is i, then the interest earned on an original principal of P in one year will be Pi.

This, added to the original principal, yields $P(1+i)$ as the accumulated amount at the end of the first year. This is also the principal at the beginning of the second year.



Accumulated Amount of \$1.00 Invested at 3% Compound Interest

Fig. 1

The interest earned during this second year will be $P(1+i)i$, and hence the accumulated amount at the end of the year will be

$$P(1+i)+P(1+i)i=P(1+i)\ (1+i)$$
$$=P(1+i)^2.$$

Again, the interest earned during the third year will be $P(1+i)^2i$ and the accumulated amount at the end will be

$$P(1+i)^2+P(1+i)^2i=P(1+i)^2(1+i)$$
$$=P(1+i)^3.$$

In general, the accumulated amount, S, of an original principal, P, invested for n years at an interest rate, i, compounded annually is

$$S=P(1+i)^n. \qquad \text{I}$$

This "compound interest formula", as given above, is set up in a form convenient for the determination of the amount to which a principal invested now would accumulate at the end of n years. In life insurance work, however, it is the inverse problem which is of most frequent occurrence; that is, what is the present value, P, of a stated sum of money, S, to be paid at some specified future date (say, n years hence)? This problem may be stated in a different, but equivalent form: What sum, P, invested today would yield an accumulated amount, S, at the end of n years? When phrased this way, it is clear that this problem also involves the same equation, I, but this time it is S which is known and P which is unknown. We solve I by dividing both sides of the equation by $(1+i)^n$, yielding

$$P=\frac{S}{(1+i)^n}=S(1+i)^{-n}. \qquad \text{II}$$

It is customary in actuarial work to use the small letter, v, as a symbol

for $\frac{1}{1+i}$ or $(1+i)^{-1}$, and hence $(1+i)^{-n}$ may be written as v^n and equation II may be rewritten as

$$P = Sv^n. \qquad \textbf{III}$$

To summarize, we may say that S is the accumulated amount of a principal, P, invested at an annual interest rate, i, for a period of n years, *or that* P is the present value or discounted value of S due in n years at an annual interest rate, i. Both statements express the same basic relationship between P and S, and equations I, II, and III are completely equivalent to one another.

INTEREST TABLES

Tables of $(1+i)^n$ and v^n at 3% interest for values of n from 1 to 50, inclusive, are given in Table LXXXII (page 982). Similar tables exist for many other rates of interest, but we shall confine our problems to those involving this one rate. It was chosen because it is used at present by a large number of life-insurance companies in computing their net premium rates and legal reserves. The use of these tables may be shown in the following examples:

Illustrative Example A

Find the compound amount at the end of 12 years of \$1200 invested at 3%, compounded annually.

Here $P = \$1200$, $i = 0.03$, $n = 12$, and we wish to find S.
Then $S = \$1200\ (1.03)^{12} = \$1200\ (1.42576) = \$1710.912$,
which we round off to the nearer cent, calling our final answer \$1710.91.

Illustrative Example B

Find the present value of \$950 payable at the end of 7 years, if interest is at 3%, compounded annually.

Here $S = \$950$, $i = 0.03$, $n = 7$, and we wish to find P.
Then $P = \$950\ (1.03)^{-7}$ or $\$950\ v^7$, where $v = (1.03)^{-1}$.
Finding v^7 in the table, we have

$$P = \$950\ (0.813092) = \$772.4374,$$

which we round off to the nearer cent, calling our answer \$772.44. (Note that in this case the *nearer* cent was the one higher, whereas in Example A it happened to be the one lower.)

TEST YOUR KNOWLEDGE OF COMPOUND INTEREST

1 Accumulate \$1000 for 25 years at 3%, compounded annually.

2 Find the present value of \$2000 due at the end of 30 years if money is worth 3%, compounded annually.

3 Jones buys a lot for \$900, paying \$150 cash. What single payment at the end of 6 years would settle the balance, if interest is at 0.03, compounded annually?

4 Smith buys some property, making a cash payment of \$1000 and agreeing to pay an additional \$6000 in 4 years. If money is worth 0.03, compounded once a year, what is the present value of the property?

Annuity certain

In general, an annuity may be said to consist of a sequence of payments, usually of equal size, made periodically. If the payments are contingent upon the occurrence of some event or series of events, the annuity is called a *contingent annuity*. The best example of this type is a life annuity, which will be discussed in detail on page 906 to 914. If, on the other hand, it is assumed that each scheduled payment over a fixed term of years is certain to be made at the time it is due, the annuity is called an *annuity certain*. Examples of this latter type are the weekly time payments on a suit of clothes, the monthly mortgage payments on a home, and the semi-annual interest payments on a corporation bond.

The time interval between payments of an annuity certain is called the *payment interval*. It may theoretically be of any length, but in practice is seldom longer than one year. Payments are most often made either monthly, quarterly, semi-annually, or annually. We shall restrict the following discussion to the case where payments are made annually and on the same day each year as that on which interest is compounded.

It is often necessary to find the *present value* of an annuity, by which we mean the sum of the values of each of the payments discounted to the beginning of the term of the annuity. For example, if a debt were to be repaid by means of a sequence of future payments, the sum of the present values of the payments, each discounted at the agreed-upon rate of interest, should equal the debt.

When one is interested in accumulating a certain amount by means of equal periodic deposits, it is the *amount* of the annuity which is involved—that is, the sum of all of the payments, each accumulated to the end of the term of the annuity. Since the present value and the amount of an annuity represent the value of exactly the same set of payments taken at different times, the present value is equal to the amount discounted from the end of the term of the annuity to the beginning of the term.

AMOUNT OF ANNUITY IMMEDIATE

Let us consider an annuity immediate with payments of \$1 made at the end of each year for a total of n years. Let $s_{\overline{n}|}$ denote the amount of this annuity taken at the end of the nth year at an interest rate of i per year. Then $s_{\overline{n}|}$ is the sum of the accumulated values of each of the payments of \$1. The first payment, being made after one year has elapsed, draws interest for $(n-1)$ years and hence is worth $(1+i)^{n-1}$ at the end of the term. Likewise, the second payment draws interest for $(n-2)$ years, the third for $(n-3)$ years, and so forth. We thus come to the next to the last payment, which, being made one year before the end of the term, accumulates to $(1+i)$, and the final payment, being

made on the last day of the term, earns no interest and hence is worth just \$1. Then we see that

$$s_{\overline{n}|}=(1+i)^{n-1}+(1+i)^{n-2}+\ \dots\ +(1+i)+1. \qquad \text{IV}$$

To simplify this expression for $s_{\overline{n}|}$, multiply both sides of equation IV by $(1+i)$, yielding

$$(1+i)s_{\overline{n}|}=(1+i)^{n}+(1+i)^{n-1}+\ \dots\ +(1+i)^{2}+(1+i). \qquad \text{V}$$

If equation IV is subtracted from equation V, we have

$$i\cdot s_{\overline{n}|}=(1+i)^{n}-1.$$

Solving this for $s_{\overline{n}|}$,

$$s_{\overline{n}|}=\frac{(1+i)^{n}-1}{i}. \qquad \text{VI}$$

An expression for $a_{\overline{n}|}$, the present value of a sequence or n payments of \$1 each, made at the end of each year, may be obtained in an analogous manner. It is easier, however, to make use of the relationship between the present value and the amount,

$$(1+i)^{-n}s_{\overline{n}|}=a_{\overline{n}|}.$$

Thus, multiplying each side of VI by $(1+i)^{-n}$ yields

$$a_{\overline{n}|}=\frac{1-(1+i)^{-n}}{i}=\frac{1-v^{n}}{i}. \qquad \text{VII}$$

The algebraic details are left as an exercise for the reader.

It should be noted carefully that $a_{\overline{n}|}$ and $s_{\overline{n}|}$ represent the present value and amount, respectively, of an annuity of n annual payments of \$1 each, the payments being made at the end of each year. In such a case, where payments are made at the end of each period, the annuity is termed an *annuity immediate.* If the payments are other than \$1, say \$$R$ each, the value of the annuity may be found by multiplying by R the corresponding value of an annuity of the same number of payments of \$1 each.

ANNUITY DUE

If payments are made at the beginning of each year instead of at the end, the annuity is called an *annuity due.* It may readily be seen that each dollar paid at the beginning of a year would accumulate to $(1+i)$ by the end of the year; hence, an annuity due of \$1 per year for n years would be exactly equivalent to an annuity immediate of $(1+i)$ dollars per year for n years. Then, if $\ddot{s}_{\overline{n}|}$ represents the amount of such an annuity due and $\ddot{a}_{\overline{n}|}$ the present value, we have

$$\ddot{s}_{\overline{n}|}=(1+i)\cdot s_{\overline{n}|} \qquad \text{VIII}$$

and

$$\ddot{a}_{\overline{n}|}=(1+i)\cdot a_{\overline{n}|}. \qquad \text{IX}$$

If the values of $s_{\overline{n}|}$ and $a_{\overline{n}|}$ from equations VI and VII are substituted

in VIII and IX, respectively, a little algebraic manipulation results in the following pair of relationships:

$$\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1, \qquad \text{X}$$

$$\ddot{a}_{\overline{n}|} = a_{\overline{n-1}|} + 1. \qquad \text{XI}$$

These derivations are left as exercises for the reader.

A statement of the values of $s_{\overline{n}|}$ and $a_{\overline{n}|}$ at 3% interest for values of n from 1 to 50 is also to be found in Table LXXXII.

Illustrative Example A

Find the present value of payments of \$250 each, made at the end of each of the next 8 years if interest is at 0.03, compounded annually.

The present value is

$$\$250 \cdot a_{\overline{8}|} = \$250\ (7.0197)$$
$$= \$1754.92.$$

Illustrative Example B

How much must be deposited at the end of each of the next 15 years in order to accumulate a fund of \$10,000 at the end of 15 years if money is worth 3%, compounded yearly?

The accumulated amount must equal $R \cdot s_{\overline{15}|}$—that is, $\$10{,}000 = R \cdot s_{\overline{15}|}$—, from which we have:

$$R = \frac{\$10{,}000}{s_{\overline{15}|}} = \frac{\$10{,}000}{18.5989}$$
$$= \$537.67.$$

Illustrative Example C

If \$50 is deposited at the beginning of each year for 5 years, what will be in the fund at the end of the five-year period if interest is at 3%, compounded annually?

Since the deposits are made at the beginning of each year, this is an annuity due; hence, the accumulated amount is

$$\$50 \cdot \ddot{s}_{\overline{5}|} = \$50\ (s_{\overline{6}|} - 1) = \$50\ (6.4684 - 1)$$
$$= \$273.42.$$

TEST YOUR KNOWLEDGE OF ANNUITIES

5 If you deposit \$100 at the end of each year for the next 7 years in a bank paying interest at 3% annually, how much will be in your account just after the last deposit?

6 A man desires to deposit with a trust company a sum which will provide his family with \$1200 at the end of each of the next 10 years. If interest on deposits is paid at the rate of 3% per year, find the sum to be deposited.

7 Johnson buys a refrigerator for \$280, paying \$30 cash and agreeing to pay off the balance in 3 annual payments. If interest is at 3%, compounded annually, find the size of these annual payments.

8 What is the present value of a sequence of payments of \$80 each, made at the beginning of each of the next 15 years, if money is worth 0.03, compounded annually?

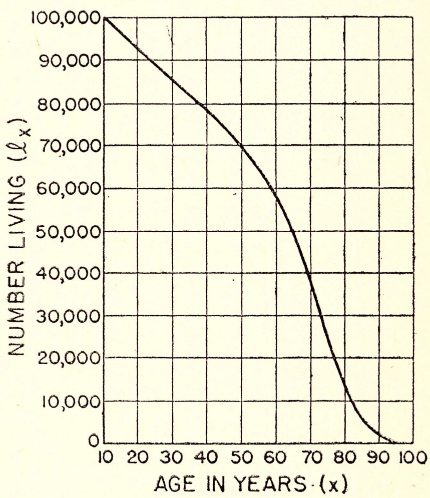
9 A man wishes to make 5 annual deposits of equal size, the first to be made today, in order to accumulate a fund of $1000 at the end of 5 years. If his deposits earn interest at 3%, compounded yearly, how large must they be?

LIFE ANNUITIES

Normally, no valid prediction can be made in regard to the time of death of an individual—in fact, if it were possible to determine one's future date of death, life insurance and life annuities, as we know them, would be both unnecessary and impossible. However, even though we cannot predict the date of death of an individual, long experience has shown that it is possible to predict with a high degree of accuracy the relative number of persons out of a given large group of persons of the same age who will die within any specified year. It is this statistical stability of large groups which makes it possible for life-insurance companies to furnish insurance protection to persons who satisfy certain requirements in regard to health and occupation at the time of application.

Mortality table

The primary basis of all actuarial theory is the mortality table, sometimes called the *life table*. (Both names are equally appropriate, since the table shows the proportion of persons who are expected to die and the proportion who are expected to survive each year of life.) The mortality table is based upon a careful study of the mortality experience over a period of time of a given group, such as the policyholders in one or a number of insurance companies, or perhaps the residents of a certain city, state, or nation. From such a study is computed the rate of mortality at each year of age, often expressed as the number of deaths expected within a year per thousand exposed. Since the mortality rate for men often differs materially from that for women of the same age, it is the usual practice to compute rates for each sex independently.



Number Living at Various Ages according to American Experience Table of Mortality
Fig. 2

Obviously, if a table is to be used in predicting future mortality on the basis of the assumption that the experience of the past will be duplicated in the future, the group of persons whose experience was used as basic data for a particular table must be as nearly like the groups for whom the table is to be used as possible. Thus, it would not do to use a table based upon census data, where persons

in all degrees of health are included, to compute insurance premiums for persons who will be medically examined before being accepted.

The mortality rates obtained directly from observed data usually exhibit minor irregularities from age to age which are attributable to pure chance. These random fluctuations are "ironed out" by a process termed *graduation*, which may be performed either graphically or by means of various mathematical formulas, and it is the graduated rates which are used in making up an actual table. A detailed consideration of graduation is beyond the scope of this article.

CONSTRUCTION OF A MORTALITY TABLE

With the rates of mortality for each age at hand, it is a comparatively simple matter to complete the mortality table. One assumes a convenient number of persons beginning with the youngest age which is to appear in the table. Usually 100,000 is taken, although other numbers are sometimes used.

In the case of the American Experience Table, upon which the actuarial tables at the end of this volume are based, we begin with 100,000 males at age 10.

As the mortality rate at this age in this table is 7.490 per thousand, there will be 749 deaths within the year out of our group of 100,000 and 99,251 survivors at age 11.

Then, if this number of survivors is multiplied by the mortality rate of 7.516 per thousand at age 11, we obtain 746 deaths and hence 98,505 survivors at age 12.

This process may be continued thus age by age until we eventually find that the last survivor of the original 100,000 dies within a year after reaching age 95.

Other mortality tables exhibit different highest ages. Such an age is, of course, an artificial convention—a table must end somewhere. It is not to be interpreted as meaning that the compiler of the table believes this "highest age" is the maximum that a human being can reach.

The American Experience Table, which was constructed about 1860, is the most widely used table in this country for legal-reserve purposes, being either prescribed or permitted by law in every state. The mortality shown at ages below 60 is considerably higher than that actually being experienced by insured persons at the present time. That this does not cause any real injustice to the younger policy holders is shown in the discussion of dividends and gross premiums in the following article.

INTERPRETATION OF A MORTALITY TABLE

A mortality table may be interpreted in more than one way. Perhaps the simplest of these is to consider it as exhibiting the entire future life history of a large group of persons, all of whom attain the

initial tabular age on the same day. In the case of the American Experience Table, this means 100,000 boys at age 10, but other initial ages occur in other tables. Under this interpretation, we may define the column headings in the following manner:

x represents the exact age in years of the individuals concerned;

l_x stands for the number of persons, out of those who begin at the initial tabular age who survive to attain precise age x;

d_x represents the number of persons out of a group of l_x persons of age x who die within a year—that is, who die before attaining age $x+1$.

q_x, which is defined as being equal to $\frac{d_x}{l_x}$, may be termed the rate of mortality at age x. By this, we mean that q_x represents the proportion of persons of exact age x who are expected to die before attaining age $x+1$. (This symbol is also often called the "probability" of death at age x.)

p_x, similarly, may be called the rate of survival or the relative frequency of survival, or the "probability" of survival at age x—that is, p_x represents the proportion of persons of exact age x who are expected to survive at least until they reach age $x+1$. In symbols, p_x is defined to be equal to $\frac{l_{x+1}}{l_x}$.

From this definition of symbols, it is clear that $d_x+l_{x+1}=l_x$—that is, out of a group of l_x persons of exact age x, there will be d_x deaths within the year and l_{x+1} survivors at the end of the year. If each side of this equation is divided by l_x, we have

$$\frac{d_x}{l_x}+\frac{l_{x+1}}{l_x}=1,$$

or

$$q_x+p_x=1. \qquad \text{XII}$$

In words, this states that the sum of the relative frequencies (or probabilities) of death and of survival, respectively, for any year of life is equal to unity.

From the mortality table, various questions regarding relative frequencies of death or survival may be answered, as is shown in the following examples:

Illustrative Examples

Assuming that the mortality in the groups to be considered will follow exactly that shown by the American Experience Table, find:—

A In a group of men aged 20, the relative frequency of surviving at least 25 years more.

B In a group of men aged 30, the relative frequency of the occurrence of death in the year following attainment of age 45.

C In a group of men aged 40, the relative frequency of death occurring between the ages of 60 and 70.

The relative frequencies required in these problems may be found in each case by dividing the number of actual occurrences (according to the mortality table) of the event in question by the total number of possible occurrences.

A The number of men in the group aged 20 is $l_{20}=92{,}637$. This is the total number of possible occurrences.

However, the table shows that only $l_{45}(=74{,}173)$ of these survived to reach age 45.

Hence, the relative frequency desired is $\dfrac{l_{45}}{l_{20}}=\dfrac{74{,}173}{92{,}637}=0.8007.$

This could also be expressed by the statement that 80.07% of men aged 20 can be expected to survive at least 25 years longer.

B The number of men who die within the year following their attainment of age 45 is shown by the table to be $d_{45}=828$. This is the number out of an initial group of $l_{30}(=85{,}441)$ who satisfy the condition of the problem.

The solution, then, is $\dfrac{d_{45}}{l_{30}}=\dfrac{828}{85{,}441}=0.0097.$

C The number of deaths occurring between ages 60 and 70 is evidently

$$l_{60}-l_{70}=57{,}917-38{,}569=19{,}348.$$

The number of men aged 40 is $l_{40}=78{,}106$. The solution is

$$\frac{l_{60}-l_{70}}{l_{40}}=\frac{19{,}348}{78{,}106}=0.2477.$$

TEST YOUR KNOWLEDGE OF THE MORTALITY TABLE

Assuming that the mortality to be experienced will follow that shown by the American Experience Table exactly, find:

10 In a group of men aged 45, the relative frequency of death occurring within 5 years;

11 In a group of men aged 23, the relative frequency of survival to age 60;

12 In a group of men aged 65, the relative frequency of death occurring between the ages of 70 and 75;

13 In a group of men aged 37, the relative frequency of the occurrence of death in the year following the attainment of age 47.

14 Given that, in a certain mortality table, $l_{20}=100{,}000$, and that q_{20}, q_{21}, q_{22}, q_{23}, and q_{24} are, respectively, equal to 3.92, 4.02, 4.12, 4.18, and 4.25 per thousand, compute, as far as this data will permit, the columns of l_x and d_x.

Life annuity

Heretofore, we have considered annuities which consist of a specified number of payments which are certain to be made. We now wish to consider annuities in which each payment is contingent upon the survival, to the payment date, of a designated individual called

the *annuitant.* Under such an annuity, payments are also ordinarily of uniform size and periodically spaced. It is usual, in elementary actuarial work to limit the discussion to life annuities involving only annual payments and to interest compounded annually.

A life annuity may be one of two kinds, a whole life annuity, under which payments continue until the death of the annuitant, and a temporary life annuity, under which payments cease at the expiration of a specified time interval or at the death of the annuitant, whichever occurs first.

ASSUMPTIONS

It is important that it be clearly understood that all net annuity and insurance calculations are based upon these assumptions:

a That the mortality to be experienced will follow *exactly* that shown in the mortality table to be used;

b That interest will be earned on all invested sums at the assumed compound interest rate;

c That no allowance is made for expenses or profits—that is, that every cent received by the insurer, plus interest earned, is ultimately returned to the group of annuitants or insured persons.

WHOLE LIFE ANNUITY

Let us assume that each of the l_x persons aged x in the mortality table applies to an insurance company for a whole life annuity immediate of $1.00 per year, first payment of $1 to be made one year hence, and offers to pay the entire net cost of this contract now. Let us assume, further, that interest will be earned by the company at 3% per year, that the mortality of the American Experience Table will be followed exactly, and that there will be no profits or expenses involved. Then the present value of the receipts from the group of annuitants must exactly balance the present value of all payments to the group.

The company will then receive a total of $l_x\, a_x$ dollars from the entire group, where a_x represents the *present value* or *single premium* or *net purchase price* of a whole life annuity immediate of $1 per year issued to a person aged x.

At the end of one year, the company will pay $1 to each of the l_{x+1} survivors, or a total of l_{x+1} dollars. Discounted to the present date, this amount has the value of $(1.03)^{-1} \cdot l_{x+1}$, which actuaries prefer to write vl_{x+1}.

At the end of the second year, the company will pay out l_{x+2} dollars, the present value of which is $(1.03)^{-2} \cdot l_{x+2}$ or $v^2\, l_{x+2}$. Similarly, the present value of the third-year payments is $v^3\, l_{x+3}$, that of the fourth-year payments is $v^4\, l_{x+4}$, and so on.

The last payments to the group will be made at age ω, where ω represents the highest age in the table at which survivors exist. In the case of the American Experience Table, $\omega=95$.

If we now equate the present value of company receipts to present value of company expenditures, we have

$$l_x a_x = v\, l_{x+1} + v^2\, l_{x+2} + v^3\, l_{x+3} + \dots + v^{\omega-x}\, l_\omega.$$

Then, solving for a_x,

$$a_x = \frac{v\, l_{x+1} + v^2\, l_{x+2} + v^3\, l_{x+3} + \dots + v^{\omega-x}\, l_\omega}{l_x}. \qquad \text{XIII}$$

Equation XIII may be used to calculate the value of a_x for any given age, x, but it involves too laborious a computation for practical use except, perhaps, when x is an advanced age.

Thus, for $x=92$ we have, using Table LXXXII for the values of v^n and Table LXXXIV for the values of l_x,

$$\begin{aligned} a_{92} &= \frac{v\, l_{93} + v^2\, l_{94} + v^3\, l_{95}}{l_{92}} \\ &= \frac{(0.970874)(79) + (0.942596)(21) + (0.915142)(3)}{216} \\ &= 0.45944. \end{aligned}$$

If x were a younger age—such as 20, for example—there would be seventy-five multiplications and one division required in the computation by this formula of the value of a_{20}. When a number of annuity values are to be computed, a considerable saving of labor may be effected by the simple device of multiplying each term in the numerator and the denominator of equation XIII by v^x. We then have

$$a_x = \frac{v^{x+1}\, l_{x+1} + v^{x+2}\, l_{x+2} + \dots + v^\omega\, l_\omega}{v^x\, l_x}.$$

It now becomes apparent that, since the exponent of v and the subscript of l agree in every individual product, it will be necessary to form only one complete set of products of the form, $v^x\, l_x$ (86 in all for the American Experience Table), to enable us to compute all of the values of a_x from $x=10$ to $x=95$ without performing any additional multiplications.

Commutation symbols—For convenience, we introduce a new symbol, D_x, to represent the product, $v^x\, l_x$. The above equation then becomes

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \dots + D_\omega}{D_x}. \qquad \text{XIV}$$

Since the computation of each whole life annuity will evidently require the summing of consecutive values of D_x, it will be convenient to prepare a table of these summations, the symbol for which is N_x. To be precise, we define

$$N_x = D_x + D_{x+1} + D_{x+2} + \dots + D_\omega.$$

Then equation XIV may be written in the very simple form,

$$a_x = \frac{N_{x+1}}{D_x}. \qquad \text{XV}$$

The symbols, D_x and N_x, and some others to be defined later, are called commutation symbols. Their values, computed at 3% interest for the American Experience Table, are to be found in Table LXXXV.*

Illustrative Example A

Find the purchase price of a whole life annuity immediate paying $600 per year to a man now aged 60.

The present value is

$$\begin{aligned}\$600a_{60} &= \$600 \cdot \frac{N_{61}}{D_{60}}\\ &= \frac{\$600\ (102{,}656)}{9{,}830.43}\\ &= \$6{,}265.61.\end{aligned}$$

Illustrative Example B

A man aged 50 buys a whole life annuity for $10,000. How much will he receive at the end of each year for life?

Let R be the unknown annual payment. Then, from XV,

$$\$10{,}000 = Ra_{50} = R \cdot \frac{N_{51}}{D_{50}}.$$

Solving for R,

$$\begin{aligned}R &= \$10{,}000\,\frac{D_{50}}{N_{51}} = \$10{,}000 \cdot \frac{15{,}922.8}{227{,}233}\\ &= \$700.73.\end{aligned}$$

TEST YOUR KNOWLEDGE OF WHOLE LIFE ANNUITIES

15 Using equation XIII, find the present value of a whole life annuity immediate of $2000 per year payable to a man now aged 90.

16 Compute the net single premium for a whole life annuity immediate of $600 per year purchased by a man aged 45.

17 A man aged 65 purchases a whole life annuity for a net single premium of $5000. What income does he receive at the end of each year from this annuity?

18 A teacher retires at age 70 on an annual pension, payable at the end of each year, of $1200. What is the present value of this pension?

WHOLE LIFE ANNUITY DUE

In the case of a whole life annuity due, the first payment is made at once and the remaining payments are made annually thereafter

* The symbols, used throughout this and the following article are standard actuarial symbols, most of which have been officially adopted by the International Congress of Actuaries. The reason for the adoption of the seemingly odd-appearing N_x is that originally the symbol, N_x, was assigned to a slightly different function and when the newer definition, as given above, was introduced, it seemed desirable to retain the letter, N, but to make it distinguishable from the one already used; hence, the adoption of the "open bar N". The older symbol, N_x, gradually became obsolete until now it is almost never used in this country.

during the lifetime of the annuitant. The symbol for the present value or net single premium of a whole life annuity due of \$1 per year issued to a man aged x is a_x. Its value can be determined as before by assuming that each of the l_x persons aged x in the mortality table purchases a whole life annuity due at the same time. Then, equating the present value of income and payments, we have

$$l_x \mathrm{a}_x = l_x + v l_{x+1} + v^2 l_{x+2} + \ldots + v^{\omega - x} l_\omega.$$

Then

$$v^x l_x \mathrm{a}_x = v^x l_x + v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + \ldots + v^\omega l_\omega.$$

This equation becomes, after each term of the form, $v^x l_x$, is replaced by the corresponding commutation symbol, D_x,

$$\mathrm{a}_x = \frac{D_x + D_{x+1} + D_{x+2} + \ldots + D_\omega}{D_x}, \qquad \text{XVI}$$

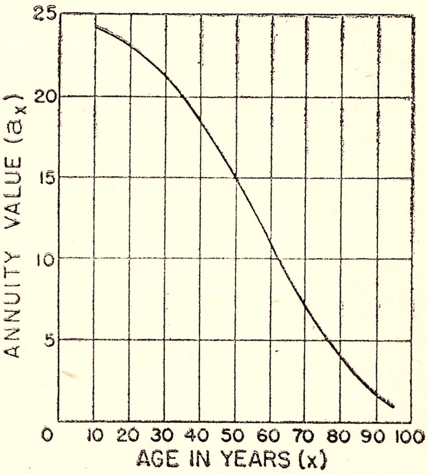
and this may be expressed more simply as

$$\mathrm{a}_x = \frac{N_x}{D_x}. \qquad \text{XVII}$$

From equation XVI we see that

$$\mathrm{a}_x = \frac{D_x}{D_x} + \frac{D_{x+1} + D_{x+2} + \ldots + D_\omega}{D_x},$$

so that $\mathrm{a}_x = 1 + a_x$. XVIII



Present Value of Life Annuity Due of \$1.00 per Year according to American Experience Table at 3% Interest

Fig. 3

This last relationship is quite useful and important. It may easily be seen to be true by noting that the only difference between a whole life annuity immediate and a whole life annuity due is the payment made at the beginning of the first year under the latter annuity. The remaining payments will coincide. Thus, the purchaser of an annuity due receives, in effect, a payment made at once plus an annuity immediate.

DEFERRED WHOLE LIFE ANNUITY

The present value of a deferred whole life annuity due of \$1 per year, under which the first payment occurs k years after the date of the contract, may be found in much the same manner as the preceding annuity values. Thus, if ${}_{k|}\mathrm{a}_x$ is the net single premium or present value of such an annuity issued to a person aged x,

$${}_{k|}\mathrm{a}_x \cdot l_x = v^k l_{x+k} + v^{k+1} l_{x+k+1} + \ldots + v^{\omega + x} l_\omega,$$

and if we divide by l_x and introduce the factor, v^x, in both numerator and denominator,

$${}_{k|}\mathrm{a}_x = \frac{D_{x+k} + D_{x+k+1} + \ldots + D_\omega}{D_x}; \qquad \text{XIX}$$

$${}_{k|}\mathrm{a}_x = \frac{N_{x+k}}{D_x}. \qquad \text{XX}$$

It should be noted that an annuity issued at age x, under which the first payment occurs at age $x+k$, can be considered as either a k-year deferred annuity due or as a (k-1)-year deferred annuity immediate.

Illustrative Example A

A man aged 57 is to receive annual payments for life of \$500 each, the first to be paid immediately. Find the present value of these payments.

This is an annuity due. The present value is

$$\$500\ddot{a}_{57}=\$500\cdot\frac{N_{57}}{D_{57}}=\$500\cdot\frac{145{,}331}{11{,}518.5}=\$6{,}308.59.$$

Illustrative Example B

An individual aged 30 pays \$5000 for a whole life annuity under which the first payment is to be made when he reaches age 65. What will be the size of the annuity payments?

Here the present value of this deferred annuity is \$5000, which must be equal to the present value of the payments. We then have

$$\$5000=R\cdot{}_{35|}\ddot{a}_{30}=R\cdot\frac{N_{65}}{D_{30}}$$

Then

$$R=\$5000\cdot\frac{D_{30}}{N_{65}}=\$5000\cdot\frac{35{,}200.6}{68{,}645.3}=\$2563.95.$$

TEST YOUR KNOWLEDGE OF ANNUITIES DUE AND DEFERRED ANNUITIES

19 A man aged 25 takes out a life insurance policy under the terms of which he is to pay a premium of \$18.35 at the beginning of each year for life. What single premium at age 25 would have been equivalent to all these annual premium payments?

20 What size annual payment for life, beginning at once, can be purchased by a man aged 68 for \$2500?

21 Find the present value of an annuity to a man aged 26 of payments of \$300 each, the first to be made when he reaches age 55.

22 An individual aged 35 pays \$5000 to an insurance company for an annuity, the first payment to be made at age 60. Find the annual income under this contract.

TEMPORARY LIFE ANNUITY

As previously defined, a temporary life annuity is one under which payments cease at expiration of a specified period, or at the prior death of the annuitant. The actuarial symbol for an n-year temporary annuity immediate of \$1 per year to a man aged x, first payment to be made at the end of the first year is $a_{x:\overline{n|}}$.

To evaluate this symbol, let us assume that such an annuity is issued to each of the l_x persons in the table aged x. The company then receives a total of $l_x\,a_{x:\overline{n|}}$ dollars from the group. It pays out l_{x+1} dollars to the survivors at the end of the first year, l_{x+2} dollars to those at the end of the second year, and so on until finally it pays l_{x+n} dollars to the survivors at the end of n years. It has now completely fulfilled its obligations to this group and

all payments cease. Discounting the payments to the date of contract and equating to the amount received by the company yields the equation,

$$l_x \cdot a_{x:\overline{n}|} = v l_{x+1} + v^2 l_{x+2} + \ldots + v^{n-1} l_{x+n-1} + v^n l_{x+n}.$$

In the usual manner, this may be reduced to

$$a_{x:\overline{n}|} = \frac{D_{x+1} + D_{x+2} + \ldots + D_{x+n-1} + D_{x+n}}{D_x}. \qquad \text{XXI}$$

Now, since

$$N_{x+1} = D_{x+1} + \ldots + D_{x+n-1} + D_{x+n} + D_{x+n+1} + \ldots + D_\omega,$$
$$N_{x+n+1} = D_{x+n+1} + \ldots + D_\omega,$$

we see that

$$N_{x+1} - N_{x+n+1} = D_{x+1} + \ldots + D_{x+n},$$

which is precisely the numerator in the right member of equation XXI. This equation may then be written in the simpler form,

$$a_{x:\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x}. \qquad \text{XXII}$$

It is left as an exercise for the reader to show that the present value of an n-year temporary life annuity due for $1 per year issued to a person aged x is

$$\ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}, \qquad \text{XXIII}$$

and the k-year deferred, n-year temporary life annuity due for $1 per year issued to a person aged x is

$$_{k|}\ddot{a}_{x:\overline{n}|} = \frac{N_{x+k} - N_{x+k+n}}{D_x}. \qquad \text{XXIV}$$

Illustrative Example A

Find the present value of a 10-year temporary annuity due paying $1000 per year to a man aged 40.

Present value is $\$1000 \cdot \frac{N_{40} - N_{50}}{D_{40}}$

$$= \$1000 \cdot \frac{444{,}394 - 243{,}156}{23{,}943.9}$$
$$= \$8404.56.$$

Illustrative Example B

What annual income, payable as a temporary life annuity at the end of each of the next 7 years to a man now aged 21, can be purchased for a single premium of $3500?

This is an annuity immediate; hence,

$$\$3500 = R \cdot \frac{N_{22} - N_{29}}{D_{21}}.$$

Solving for the unknown payment, we have

$$R = \frac{\$3500\, D_{21}}{N_{22} - N_{29}}$$
$$= \frac{\$3500\,(49{,}408.3)}{1{,}077{,}510 - 779{,}046}$$
$$= \$579.40.$$

TEST YOUR KNOWLEDGE OF TEMPORARY LIFE ANNUITIES

23 A certain life insurance policy requires the payment of premiums of \$33.50 at the beginning of each year for 20 years by an individual now aged 36. Find the present value of these premiums.

24 Find the present value of a 30-year temporary life annuity immediate of \$100 per year issued to a man aged 60.

25 What annual payment, payable at the end of each of the next 15 years during the continued survival of an annuitant aged 43, can be purchased for a net single premium of \$7500?

26 On his son's fourteenth birthday, a father pays \$3000 to an insurance company for 4-year temporary annuity, first payment to be made to the son on his eighteenth birthday, to provide for his college expenses. Find the size of these annual payments.

GENERAL ANNUITY FORMULA

At this point, it might appear to the reader as if many different formulas, difficult to memorize, have been developed. Actually, these annuity formulas are so closely related that a single unifying formula may be developed to cover all life annuities having annual payments. Consider the following tabulation of the present-value formulas already derived:

Type of Annuity	Age at Date of Present Value	Age at Date of First Payment	Term in Years	Age at Date of Last Payment	Formula for Present Value
Whole life:					
Annuity immediate	x	$x+1$	life	ω	$\frac{N_{x+1}}{D_x}$
Annuity due	x	x	life	ω	$\frac{N_x}{D_x}$
k-year deferred annuity due	x	$x+k$	life	ω	$\frac{N_{x+k}}{D_x}$
n-year temporary:					
Annuity immediate	x	$x+1$	n	$x+n$	$\frac{N_{x+1}-N_{x+n+1}}{D_x}$
Annuity due	x	x	n	$x+n-1$	$\frac{N_x-N_{x+n}}{D_x}$
k-year deferred annuity due	x	$x+k$	n	$x+k+n-1$	$\frac{N_{x+k}-N_{x+k+n}}{D_x}$

The reader will readily observe that, in each case, the subscript on the D-symbol in the denominator of the present-value formula agrees with the age at the date at which the annuity is to be valued. Next, it is apparent that the subscript on the first N is equal to the age at which the first annuity payment is to be made. In the case of the temporary annuities, there is a second N-term in the numerator which has a subscript equal to the age one year after the last annuity payment has been made—or, it may be said that this last subscript is equal to n plus the subscript on the first N. Hence, we may express

the latter three temporary annuity formulas by means of the expression,

$$\frac{N_e - N_f}{D_g}, \qquad \text{XXV}$$

where e is the age at which the first annuity payment is made; f is the age one year after the last annuity payment is made, or $f=e+n$; and g is the age at the date on which the value of the annuity is to be determined.

This formula also represents the present value of the various whole life annuities, since, in these cases, the value of f is $\omega+1$, payments being made right up to age ω, the last age at which survivors exist according to the mortality table. In this case, since all values of l_x, and hence D_x and N_x, are 0 for values of $x>\omega$, the second term in the numerator vanishes. Hence, formula XXV may be used to cover all types of annuities with annual payments, and is the only formula that need be memorized.

The following examples show the use of formula XXV:

Illustrative Examples

Express in commutation symbols the formulas for the present value of each of the following annuities:

A A 12-year temporary life annuity due issued to a man aged 50.

Here, $e=50, f=50+12=62, g=50$, and the formula is $\frac{N_{50}-N_{62}}{D_{50}}$.

B A 25-year temporary annuity immediate issued at age 40.

In this case, $e=41, f=41+25=66, g=40$.

The present value is $\frac{N_{41}-N_{66}}{D_{40}}$.

C A whole life annuity immediate issued at age 35.

Here $e=36$, $f=96$, and $g=35$.

The formula, then, is $\frac{N_{36}-N_{96}}{D_{35}}$, which, since $N_{96}=0$, reduces to $\frac{N_{36}}{D_{35}}$.

D A 20-year temporary life annuity due deferred 10 years, issued to a man aged 25.

In this case, $e=35, f=35+20=55, g=25$.

The formula for the present value of this annuity is $\frac{N_{35}-N_{55}}{D_{25}}$.

Pure endowment

It is often useful in actuarial work to find the present value to an individual now aged x of a single payment to be made at the end of a specified period provided the individual is then alive; otherwise, no payment is made. Such a future contingent payment is called a pure endowment.

Pure endowments are never sold separately by insurance companies, but only in combination with various forms of life insurance. A pure

endowment is actually the very antithesis of life insurance, since in this case the persons who die lose what they have contributed while those who survive are financially rewarded.

To develop an expression in commutation symbols for the present value of a k-year pure endowment of \$1, assume that each of a group of l_x persons aged x agrees to pay ${}_kE_x$ dollars for the benefit of receiving a single payment of \$1 at the end of k years, provided that the person is alive at that time. Then the company would receive $l_x \cdot {}_kE_x$ dollars now and would pay out l_{x+k} dollars to the survivors of the original group at the end of k years. This sum, discounted to the present date, is now worth $v^k l_{x+k}$. Equating the present value of receipts and expenditures by the company, we have

$$l_x \cdot {}_kE_x = v^k l_{x+k},$$

$${}_kE_x = \frac{v^k l_{x+k}}{l_x} = \frac{v^{x+k} l_{x+k}}{v^x l_x},$$

$${}_kE_x = \frac{D_{x+k}}{D_x}. \qquad \text{XXVI}$$

The relationship between pure endowment and life annuities is very close. A k-year pure endowment can be considered as a k-year deferred, one-year temporary life annuity due.

Formula XXVI thus follows as a special case of Formula XXV, where

$$e = x+k,\ f = x+k+1,\ g = x.$$

In this case, we have

$$\frac{N_{x+k} - N_{x+k+1}}{D_x},$$

but, since the numerator is readily seen to equal D_{x+k}, the equivalence of the formulas is demonstrated.

On the other hand, each individual payment in any life annuity may be considered as a pure endowment, so that the present value of any life annuity is merely the sum of a sequence of pure endowments. For example, for a whole life annuity immediate of \$1 per year, issued to a man aged x, the present value may be written

$$a_x = {}_1E_x + {}_2E_x + {}_3E_x + {}_4E_x + {}_5E_x + \cdots + {}_{\omega-x-1}E_x + {}_{\omega-x}E_x,$$

which becomes, after we substitute the value in commutation symbols of the pure endowments,

$$a_x = \frac{D_{x+1}}{D_x} + \frac{D_{x+2}}{D_x} + \cdots + \frac{D_{\omega-1}}{D_x} + \frac{D_\omega}{D_x}$$

$$= \frac{D_{x+1} + D_{x+2} + \cdots + D_{\omega-1} + D_\omega}{D_x} = \frac{N_{x+1}}{D_x}.$$

Foreborne annuity

In the preceding discussion of life annuities, we were concerned entirely with the problem of finding the present value of a set of contingent payments. This was done by finding the present value of the entire sequence of payments made to the successive survivors of a group and dividing this sum by the number in the original group.

We thus obtained an average present value per original group member.

This same sort of process may be used to find the average value of such a sequence of contingent payments at some other date than that of the beginning of the contract—in particular, the value at the termination date. This is a slightly more complicated problem than that of finding the amount of an annuity certain because here we have survivorship as well as interest to consider. The idea may best be understood by considering an example:

Illustrative Example

Suppose that l_{25} persons aged 25 agree to pay \$1 each into a fund at the beginning of each year for the next 10 years, and that, at the end of that time, they will divide the accumulated fund among the survivors. The share of each survivor will be called the value of a 10-year foreborne annuity of \$1 per year issued at age 25, and its symbol is ${}_{10}u_{25}$. As before, we shall assume that the American Experience tabular mortality will be followed precisely and that interest will be earned at 3%.

In this case, we find that the fund receives a deposit of l_{25} dollars at the beginning, and this sum will have accumulated to $(1.03)^{10}l_{25}$ at the end of the 10-year term. The deposits at the beginning of the second year total l_{26} dollars, which will have accumulated to $(1.03)^9 l_{26}$ dollars at the end of the term, and so on for the remaining deposits. The last one, made just one year before the termination date, would total l_{34} dollars and will have accumulated to $(1.03)l_{34}$ dollars at the end of the term. Then the accumulated value of the deposits, which, by agreement, is the property of the l_{35} survivors, is equal to

$$l_{35}\,{}_{10}u_{25}=(1.03)^{10}l_{25}+(1.03)^9 l_{26}+\ .\ .\ .\ +(1.03)l_{34}.$$

Solving for ${}_{10}u_{25}$,

$${}_{10}u_{25}=\frac{(1.03)^{10}l_{25}+(1.03)^9 l_{26}+\ .\ .\ .\ +(1.03)l_{34}}{l_{35}}.$$

Now multiply both numerator and denominator of the right member by v^{35}—that is, by $(1.03)^{-35}$. Then

$${}_{10}u_{25}=\frac{(1.03)^{-25}l_{25}+(1.03)^{-26}l_{26}+\ .\ .\ .\ +(1.03)^{-34}l_{34}}{v^{35}l_{35}},$$

or

$${}_{10}u_{25}=\frac{v^{25}l_{25}+v^{26}l_{26}+\ .\ .\ .\ +v^{34}l_{34}}{v^{35}l_{35}},$$

which may be written as

$${}_{10}u_{25}=\frac{D_{25}+D_{26}+D_{27}+\ .\ .\ .\ +D_{34}}{D_{35}},$$

and finally as

$${}_{10}u_{25}=\frac{N_{25}-N_{35}}{D_{35}}.$$

A generalization of this special case yields the formula for the value of an n-year foreborne annuity issued at age x:

$${}_{n}u_{x}=\frac{N_x-N_{x+n}}{D_{x+n}}. \qquad \text{XXVII}$$

This annuity is sometimes referred to as the accumulated value of the individual survivor's payments with the benefit of interest and survivorship.

It is interesting to note that the general annuity formula which we derived to represent the present value of all life annuities with annual payments may also be interpreted in a way to give the value of a foreborne annuity. We defined g to be the age on the date at which the value of the annuity was to be determined; in the case of a foreborne annuity, this age is that at the end of the term—that is, $g=f=e+n$. (See formulas XXV and XXVII.)

The following set of miscellaneous problems is designed to test the reader's mastery of the entire section on life annuities, including pure endowments and foreborne annuities.

TEST YOUR KNOWLEDGE OF LIFE ANNUITIES BY THESE EXERCISES

27 A man aged 68 buys a life annuity for $3500. What amount will he receive at the end of each year for life?

28 Find the present value of a pure endowment of $500 to be paid in 10 years to a man now aged 44, if he is then alive.

29 What must a man now aged 29 pay for a life annuity of $1800 per year, first payment to be made at age 70?

30 What sized annual payment can be purchased for a single premium of $1000, if the payments are to begin at once and continue for 10 payments during the survival of an annuitant now aged 53?

31 A group of men now aged 36 mutually agree to deposit $100 at the beginning of each year for the next 8 years and at the end of that time the survivors are to divide the fund. How much will each survivor receive?

32 What sized payment, made at the end of 12 years to an individual now aged 48 if he is alive at that time, can be had for an immediate payment of $2000?

33 Find the present value of a 7-year deferred, 14-year temporary annuity due of $300 per year issued to a man aged 63.

34 A man aged 35 wishes to provide for a payment of $25,000 at age 65 if he is then alive. What annual payment must he deposit at the beginning of each year for the next 30 years to pay for this benefit if the contract provides that no return of deposits will be made if he dies before reaching 65? (*Hint*: His accumulation at age 65 is the accumulated value of a foreborne annuity.)

35 A group of men now aged 21 agree to deposit $50 at the beginning of each year for the next 10 years and then to allow the fund to increase at interest until it is divided among the survivors at age 65. How much will each survivor receive? (*Hint*: The accumulation of the foreborne annuity at age 31 is the present value of the pure endowment due at age 65, 34 years later.)

36 How much must each member of a group of men aged 27 deposit at the beginning of each year for the next 20 years if each of the survivors is to receive $10,000 at age 60 when the fund is to be divided equally among them? What would be the size of the deposits if the estates of those who died before reaching age 60 were also to share in the division? What would be the size if these estates were also required to continue making deposits just as if the individual were still alive?

LIFE INSURANCE NET PREMIUMS As has been pointed out previously, life insurance is possible on a scientific basis only when a large group of persons is insured under one organization. In such instances, if reasonable care has been exercised in the selection of the insured persons so that they constitute the kind of group to which the assumed mortality table is applicable, the mortality to be experienced may be predicted with considerable accuracy under normal conditions. Under these assumptions, it is possible to calculate the amount which the various individual insured persons must pay in order to provide the desired benefits to be paid at the death of each. Obviously, wars, epidemics, or other catastrophies might conceivably invalidate predictions of any kind. In actual practice, prudent management provides for margins of safety so that almost any probable deviation from the predicted mortality can be withstood.

We shall consider only the types of life insurance issued by so-called "old line" or *legal reserve* life insurance companies. At the present time, this sort of scientific insurance is also issued by a number of fraternal insurance associations.

The written contract embodying the terms of the insurance agreement is called the *policy*, and the insured person is called the *policyholder* or the *insured*. The amount to be paid by the company, or *insurer*, is called the *benefit*, and the person to whom the benefit is to be paid in the event of the death of the insured is called the *beneficiary*. In certain instances, the benefit is paid to the policyholder himself, during his lifetime.

In return for the contracted benefit, the policyholder agrees to make to the company one or more payments, called *gross premiums*. These premiums are, of course, payable only during the lifetime of the policyholder. The date on which the contract is entered into is called the *date of issue* or *policy date* and each anniversary of this date is called a *policy anniversary*. The successive years after the date of issue are called *policy years*. Thus, the first twelve months following the issuance of an insurance policy constitute the first policy year; immediately after the first policy anniversary, the insured has entered his second policy year. The *age* of the insured at the date of issue and at each succeeding policy anniversary is taken to be that *at his birthday nearest to the date in question*.

Types of insurance

There are three basic types of insurance: *whole life*, *term*, and *endowment insurance*. Sometimes policies are devised which consist of various combinations of these three basic types, but they can always be analyzed into their component parts.

Under a *whole life* policy, the company is obligated to pay the death

benefit to a selected beneficiary at the death of the insured regardless of when it may occur.

In the case of a *term* policy, the death benefit is due only if the insured dies during a specified period of years, and the policy becomes valueless if the insured survives this period.

An *endowment insurance* policy provides for the payment of the *face amount*, or amount stated in the contract, at the expiration of a specified term of years if the insured is then alive or at his death if it occurs prior to that time.

ASSUMPTIONS

The premium which should be paid for the benefits provided in any policy may be readily calculated on the basis of the same three assumptions made in the case of life annuities, plus one additional assumption. These are:

That the mortality experience will follow the mortality table exactly;
That interest will be earned at the specified rate;
That no overhead or other expenses of any kind are involved; and
That all death benefits will be paid *at the end* of the policy year in which death occurs.

Under these assumptions, the present value at the date of issue of the payments to be made to the company by a group of policyholders must exactly balance the present value on the same date of the payments to be made by the company to the group. A premium computed on such a basis is called a *net premium.*

In actual practice, of course, expenses do exist and must be provided for, and allowances must be made for contingencies of various kinds, so that the premiums actually paid by the insured, called *gross premiums*, will be greater than the corresponding net premiums. The difference between net and gross premiums is called the *loading.*

In this section, we shall consider only net premiums, assuming an interest rate of 3% and a mortality equal to that of the American Experience Table. As stated in the preceding matter, we shall also assume that death claims are paid at the end of the year in which death occurs. This assumption is not in strict accordance with the facts, as for many years American insurance companies have paid death claims immediately upon receipt of proof of death, though they still compute net premiums on the basis of this assumption.

WHOLE LIFE INSURANCE

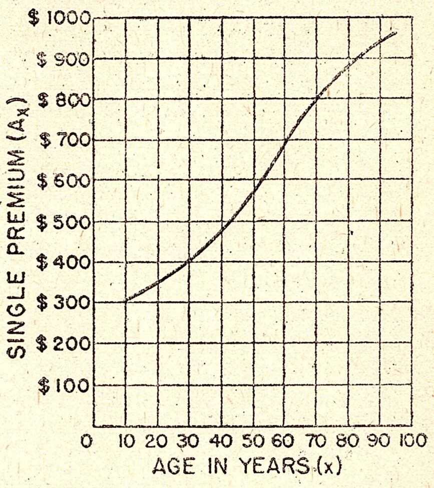
Single premium—Under the terms of a whole life policy for $1 issued to a person aged x, the company agrees to pay $1 at the end of the policy year in which the insured dies, regardless of when that death may occur. The present value of this benefit is called the *net single premium* and is the amount which the insured would pay for the benefit if he agreed to discharge his entire obligation to the company

in one installment at the policy date. Such single premium policies are actually sold, but not in large numbers.

To compute the net single premium for a whole life policy for \$1, we assume as before that each of the l_x persons alive at age x according to the mortality table purchases such a policy on the same day. The company receives a total of $l_x A_x$ dollars from the group on the date of issue, where A_x is the symbol for the net single premium.

At the end of the first policy year, the company pays \$1 to each of the beneficiaries of the d_x persons who die during the year. Discounted to the date of issue, these death benefits have a present value of $v \cdot d_x$.

At the end of the second policy year, the company pays \$1 for each of the d_{x+1} persons who dies during that year. Discounted to the present, this amounts to $v^2 d_{x+1}$. If this process is carried out to the highest age existent in the table and the present value of the company's receipts is equated to the present value of its payments, we have



Net Single Premium for Whole Life Insurance for \$1000 (American Experience Table: 3%)

Fig. 4

$$l_x A_x = v\,d_x + v^2 d_{x+1} + v^3 d_{x+2} + \ . \ . \ . \ + v^{\omega-x+1} d_\omega. \qquad \text{XXVIII}$$

As in the case of life annuities, we divide both members of equation XXVIII by l_x and then multiply numerator and denominator of the resulting right member by v^x. This yields

$$A_x = \frac{v^{x+1} d_x + v^{x+2} d_{x+1} + \ . \ . \ . \ + v^{\omega+1} d_\omega}{v^x l_x}.$$

We note that this formula is quite analagous to the expression obtained for a_x. It may be simplified in appearance by defining two new commutation symbols. First, let $C_x = v^{x+1} d_x$; then

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \ . \ . \ . \ + C_\omega}{D_x}. \qquad \text{XXIX}$$

If we now define $M_x = C_x + C_{x+1} + C_{x+2} + \ . \ . \ . \ + C_\omega$, the final form for the expression for A_x becomes

$$A_x = \frac{M_x}{D_x} \qquad \text{XXX}$$

Annual premiums—Few persons care to purchase single premium policies. It is much more common to pay for insurance benefits by means of equal annual premium payments made either throughout the entire length of the policy or for a shorter period. Since such annual insurance premiums are payments of equal size, made at the beginning of each year only in the event of the survival of the insured to the

payment date, they constitute a *life annuity due.* (The payments are contingent in this case upon the survival of the payor rather than the payee, but this difference has no effect upon the mathematical computation of present value.)

When a whole life policy is paid for by annual premiums payable throughout life, the resultant type of insurance is called *ordinary life*, and is the most popular of all forms of life insurance.

Let us find the annual premium, P_x, for an ordinary life policy issued at age x. The insured, if he were to pay for the benefit in one payment, would pay A_x at the date of issue. Instead, he seeks to substitute a life annuity due of P_x dollars per year for this net single premium. Hence,

$$P_x \text{a}_x = A_x,$$

or

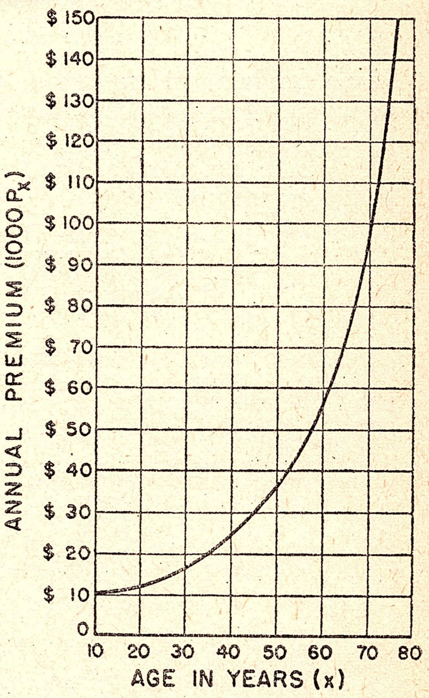
$$P_x = \frac{A_x}{\text{a}_x}. \qquad \text{XXXI}$$

The value of P_x may be expressed in commutation symbols as

$$P_x = \frac{M_x}{D_x} \div \frac{N_x}{D_x},$$

$$P_x = \frac{M_x}{N_x}. \qquad \text{XXXII}$$

This value could have been obtained by the same method used so frequently heretofore—that is, by equating the discounted values of all present and future payments *to* the company to the discounted value of all future payments *by* the company. The reader will find it instructive to go through this derivation.



Net Annual Premiums for $1000 of Ordinary Life Insurance (American Experience Table, 3%)

Fig. 5

Limited payment policies — If the insured elects to pay his obligation by means of a limited number of premium payments (say, m payments), these constitute an m-year temporary life annuity due whose value at the policy date is ${}_mP_x \cdot \text{a}_{x:\overline{m|}}$, where ${}_mP_x$ is the size of the premium for an m-payment whole life insurance policy for $1 issued to a man aged x. We then have

$${}_mP_x \cdot \text{a}_{x:\overline{m|}} = A_x,$$

from which

$${}_mP_x = \frac{A_x}{\text{a}_{x:\overline{m|}}}, \qquad \text{XXXIII}$$

and, in commutation symbols,

$${}_mP_x = \frac{M_x}{N_x - N_{x+m}}. \qquad \text{XXXIV}$$

Values of C_x and M_x are to be found in Table LXXXV with the values of D_x and N_x used previously.

All insurance and annuity computations can be made with the aid of the commutation columns alone. However, for more rapid computation in later and more involved problems, it is usual to use some derived tables, some of which are included in this volume. Thus, values of a_x and A_x for all ages are to be found in Table LXXXIII. (It is recommended that, at this stage, no use be made by the reader of any derived table until he is thoroughly familiar, by actual practice, with the methods for finding annuity values and insurance premiums by the use of commutation columns alone.)

Illustrative Example A

Using fundamental equation XXVIII, find the net single premium for a whole life insurance policy of \$1000 on a life aged 93.

The single premium is equal to

$$\$1000\, A_{93} = \$1000 \cdot \frac{(1.03)^{-1} d_{93} + (1.03)^{-2} d_{94} + (1.03)^{-3} d_{95}}{l_{93}}$$

$$= \$1000 \cdot \frac{(0.970874)\ (58) + (0.942596)\ (18) + (0.915142)(3)}{79}$$

$$= \$962.31.$$

Illustrative Example B

Find the net single premium for a whole life insurance policy for \$5000 issued to a man aged 30.

The net single premium is

$$\$5000\, A_{30} = \$5000 \cdot \frac{M_{30}}{D_{30}}$$

$$= \$5000 \cdot \frac{13{,}574.8}{35{,}200.6}$$

$$= \$1928.21.$$

Illustrative Example C

What amount of whole life insurance could be bought for a single premium of \$1000 by a man aged 50?

Let F equal the face amount of the policy—that is, the amount of death benefit which the policy promises to pay. Then

$$\$1000 = F \cdot A_{50} = F \cdot \frac{M_{50}}{D_{50}},$$

or

$$F = \$1000 \cdot \frac{D_{50}}{M_{50}}$$

$$= \$1000 \cdot \frac{15{,}922.8}{8{,}840.57}$$

$$= \$1801.11.$$

Illustrative Example D

Find the net annual premium for an ordinary life policy for $1000 issued at age 21.

The net annual premium is

$$\$1000\, P_{21} = \$1000 \cdot \frac{M_{21}}{N_{21}}$$
$$= \$1000 \cdot \frac{16{,}585.4}{1{,}126{,}919}$$
$$= \$14.72.$$

Illustrative Example E

Find the net annual premium for a twenty-payment life policy for $2000 issued at age 29.

The net annual premium is

$$\$2000 \cdot {}_{20}P_{29} = \$2000 \cdot \frac{M_{29}}{N_{29} - N_{49}}$$
$$= \$2000 \cdot \frac{13{,}871.0}{779{,}046 - 259{,}774}$$
$$= \$53.42.$$

TEST YOUR KNOWLEDGE OF WHOLE LIFE NET PREMIUMS

37 Find the net single premium for a whole life insurance policy of $1000 on the life of a man (a) aged 25; (b) aged 55; (c) aged 85.

38 Using equation XXVIII, find the net single premium on a whole life insurance policy for $2500 to a man aged 92.

39 How much whole life insurance can a man aged 42 purchase for a net single premium of $500?

40 Find to the nearest cent the net annual premium for each of the following policies of $1000 each issued at age 29: (a) ordinary life; (b) ten-payment life; (c) thirty-payment life.

41 What amount of twenty-payment life insurance could be purchased for an annual premium of $100 by a man aged 21?

TERM INSURANCE

An insured under a term insurance policy is protected for only a limited period—that is, if his death occurs during the specified period, his beneficiary will receive the face amount of the policy but, if he survives the period, the policy expires without value.

Sometimes a term policy carries with it an extra privilege in the form of the right to *convert* the policy into another form of insurance (*e.g.*, ordinary life), without a medical examination, before the term policy expires. Some term policies are *renewable*; this means that at the expiration date the insured may take out another policy of the same kind and amount without taking a medical examination. Often

these two features are combined so that some term policies are both renewable and convertible.

It is obvious that it is costly for the company to offer these features, since persons whose health fails during the term period and who thus could not otherwise obtain life insurance protection will almost surely avail themselves of these privileges. On the other hand, persons in excellent health at the expiration of the term policy find the privileges of little value since they can obtain insurance anywhere by passing a medical examination, and some are very apt to insure in other companies. Thus, there is almost sure to be a certain amount of "selection against the company". A term policy containing one or more of these features must bear a higher premium than a regular term policy without the features. The computation of such a higher premium is too complicated to be considered here.

Net single premium—Let us derive an expression in commutation symbols for the net single premium for an n-year term insurance policy for \$1, without any of the special features described above, issued to a person aged x. As in our preceding work, we assume that all l_x persons aged x in the life table apply for and receive such a policy. Then the company receives a total of $l_x \cdot A^1_{x:\overline{n|}}$ dollars at once and pays out d_x dollars at the end of the first year, d_{x+1} dollars at the end of the second year, and so forth until the final payments totalling d_{x+n-1} dollars are made at the end of the nth year. Thereafter, no more death benefits are to be paid. If we discount each of these payments to the date of issue and equate the sum to the amount paid to the company, we have

$$l_x \cdot A^1_{x:\overline{n|}} = v\,d_x + v^2\,d_{x+1} + \;.\;.\;.\; + v^{n-1}\,d_{x+n-2} + v^n\,d_{x+n-1}. \qquad \text{XXXV}$$

Solving for $A^1_{x:\overline{n|}}$, introducing v^x, and substituting commutation symbols as before, yields

$$A^1_{x:\overline{n|}} = \frac{C_x + C_{x+1} + C_{x+2} + \;.\;.\;.\; + C_{x+n-1}}{D_x}. \qquad \text{XXXVI}$$

Now, since

$$M_x = C_x + C_{x+1} + \;.\;.\;. + C_{x+n-1} + C_{x+n} + \;.\;.\;.\; + C_\omega\,,$$

and

$$M_{x+n} = \qquad\qquad C_{x+n} + \;.\;.\;.\; + C_\omega\,,$$

it is evident that the numerator in the right member of equation XXXVI may be written as $M_x - M_{x+n}$, so that we have

$$A^1_{x:\overline{n|}} = \frac{M_x - M_{x+n}}{D_x}. \qquad \text{XXXVII}$$

The net single premium for a term policy is, of course, considerably less than that for a whole life policy. Nevertheless, it is not common for even such single premium policies to be sold. As a usual thing, the insured pays for the protection offered by a term policy by means of annual premiums payable throughout the entire term of the policy. Occasionally, in the case of a term policy which extends over a long

period of time, as 20 or 30 years, the number of premiums will be less than the number of years included. For example, one may take out a twenty-payment, thirty-year term policy. We shall develop formulas for the annual premiums for these various cases.

Annual premiums—For an n-year term insurance with n premiums, these annual premiums constitute an n-year temporary life annuity due, and hence we have

$$P^{1}_{x:\overline{n}|} \cdot a_{x:\overline{n}|} = A^{1}_{x:\overline{n}|}.$$

Solving for $P^{1}_{x:}a$,

$$P^{1}_{x:\overline{n}|} = \frac{A^{1}_{x:\overline{n}|}}{a_{x:\overline{n}|}}. \qquad \text{XXXVIII}$$

In commutation symbols, equation XXXVIII becomes

$$P^{1}_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} \div \frac{N_x - N_{x+n}}{D_x},$$

$$P^{1}_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \qquad \text{XXXIX}$$

For an m-payment, n-year term insurance, the annual premium would be

$${}_mP_{x:\overline{n}|} = \frac{A^{1}_{x:\overline{n}|}}{a_{x:\overline{m}|}},$$

and, in commutation symbols,

$${}_mP^{1}_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+m}}. \qquad \text{XL}$$

Theoretically, the annual premiums could be spread out over a period of any length up to the whole of life. Practically, however, the length of the premium-paying period never extends beyond the date of expiration of the term of the insurance protection. The reason for this restriction appears obvious.

Natural premium—The net single premium for a one-year term insurance of $1 issued to a man aged x is called the natural premium at age x and is denoted by the symbol, c_x. The value of c_x expressed in commutation symbols may be readily found as a special case of equation XXXVII by letting $n=1$ and substituting C_x for the resulting binomial, $M_x - M_{x+1}$. Thus

$$c_x = \frac{C_x}{D_x}. \qquad \text{XLI}$$

The natural premium represents the cost of protection of $1 at each age. This cost increases year by year, slowly at first, but rapidly at the higher ages. For example, the natural premium at age 25 is $7.78 per $1000 while the premium for an ordinary life policy issued at that age is $15.10. However, at age 60, the insured will still be paying only $15.10 under the terms of his ordinary life contract, while the natural premium at that age is $25.79 and rising rapidly. There are some persons who believe that all life insurance should be written on the annual renewable term plan, but the public has shown a decided

preference for insurance paid for by means of level premiums rather than by premiums that increase each year.

Illustrative Example A

Find the net single premium for a five-year term insurance of $1000 on the life of a man aged 42.

The single premium is

$$\$1000\ A^{1}_{42:\overline{5|}} = \$1000 \cdot \frac{M_{42} - M_{47}}{D_{42}}$$

$$= \$1000 \cdot \frac{10{,}549.1 - 9{,}473.08}{22{,}124.7}$$

$$= \$48.63.$$

Illustrative Example B

Find the net annual premium for a twenty-payment, thirty-five-year term insurance policy for $10,000 issued to a man aged 30.

The annual premium is

$$\$10{,}000\ {}_{20}P^{1}_{30:\overline{35|}} = \$10{,}000 \cdot \frac{M_{30} - M_{65}}{N_{30} - N_{50}}$$

$$= \$10{,}000 \cdot \frac{13{,}574.8 - 5{,}224.80}{742{,}484 - 243{,}156}$$

$$= \$167.22.$$

TEST YOUR KNOWLEDGE OF TERM INSURANCE PREMIUMS

42 Find the net single premium for ten-year term policy for $3000 issued to a man aged 24.

43 How much three-year term insurance can be bought by an individual aged 51 for a net single premium of $50?

44 What is the annual premium for a fifteen-year term policy for $1000 issued to a man aged 39?

45 Find the premium for a one-year term insurance policy issued to a man aged: (a) 20; (b) 35; (c) 50; (d) 65; (e) 80.

ENDOWMENT INSURANCE

Under the terms of an endowment insurance policy, the company agrees, in the event of the death of the insured during the specified period, to pay the face amount of the policy to the beneficiary at the end of the year in which death occurs and, in the event of the survival of the insured to the date of maturity (the end of the period), to pay the face amount to him. It is evident that this policy contains an appreciable investment element as well as protection and must be more costly than the forms of insurance studied previously. Let us derive the value of the net single premium in commutation symbols.

Single premium—The company, upon insuring l_x persons aged x under single premium n-year endowment policies for a face amount of $1 each, would receive a total amount of $l_x A_{x:\overline{n|}}$ dollars. In return for this,

the company would pay out $d_x, d_{x+1}, d_{x+2}, \ldots, d_{x+n-1}$ dollars, respectively, in successive years in death claims over the n-period. In addition, it would pay out one dollar to each of the l_{x+n} survivors on the date of maturity. If we discount each payment to the date of issue and equate the present values of company receipts and disbursements, we have

$$l_x A_{x:\overline{n|}} = v\,d_x + v^2 d_{x+1} + v^3 d_{x+2} + \ldots + v^n d_{x+n-1} + v^n l_{x+n}. \qquad \text{XLII}$$

By the usual steps, this may be reduced to

$$A_{x:\overline{n|}} = \frac{C_x + C_{x+1} + \ldots + C_{x+n-1} + D_{x+n}}{D_x},$$

and then to

$$A_{x:\overline{n|}} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \qquad \text{XLIII}$$

From equations XXVI, XXXVII, and XLIII, we see that the net single premium for an n-year endowment insurance policy for \$1 may be analyzed into two portions: the single premium for an n-year pure endowment of \$1 plus the single premium for an n-year term insurance of \$1. That is,

$$A_{x:\overline{n|}} = {}_nE_x + A^{1}_{x:\overline{n|}}. \qquad \text{XLIV}$$

An analysis of the benefits as described at the beginning of this paragraph would result in the same conclusion.

Annual premiums—As in the case of term insurance, it is customary to pay for the benefits of an n-year endowment insurance policy by means of annual premiums paid throughout the entire period of the policy so that such premiums form an n-year temporary life annuity due. The value of the annual premium is found to be

$$P_{x:\overline{n|}} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}. \qquad \text{XLV}$$

Premiums may be paid over a shorter period than n years. The formula for the annual premium for an m-payment, n-year, endowment insurance policy for \$1, where m is less than n, is

$$_mP_{x:\overline{n|}} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+m}}. \qquad \text{XLVI}$$

Illustrative Example

Compute the annual premium, payable for 10 years, on a twenty-year endowment insurance for \$1000 on a man aged 30.

The premium is

$$\$1000\ {}_{10}P_{30:\overline{20|}} = \$1000 \cdot \frac{M_{30} - M_{50} + D_{50}}{N_{30} - N_{40}}$$

$$= \$1000 \cdot \frac{13{,}574.8 - 8{,}840.57 + 15{,}922.8}{742{,}484 - 444{,}394}$$

$$= \$69.30.$$

TEST YOUR KNOWLEDGE OF ENDOWMENT INSURANCE PREMIUMS

46 Compute the annual premium for a twenty-year endowment policy for \$2500 issued to a man aged 26.

47 How much single premium, thirty-year endowment insurance can be purchased for a net premium of \$2000 by a man aged 48?

48 How large an endowment policy maturing in 15 years can be purchased by an annual payment of \$100 to be paid each year for 10 years by an individual now aged 45?

DEFERRED INSURANCES

While it is unusual for an insurance policy to be sold which provides that the protection does not begin at once but only after the expiration of some stated period, the concept of such a deferred insurance is occasionally helpful in actuarial calculations, particularly those pertaining to combination policies or to modified reserves. For this reason, we shall devote some attention to various types of deferred insurances.

The single premiums for deferred whole life, term, or endowment insurances may be derived by the same method used so frequently in the preceding pages. However, it is shorter to use an alternative method here which is best shown by means of an example. Consider a ten-year term insurance for \$1, deferred for 5 years. This means that the insured is protected from the sixth year to the fifteenth, inclusive. It is evident, then, that this policy differs from a straight fifteen-year term insurance only in that it omits protection for the first 5 years; hence, it is equivalent to a fifteen-year term insurance minus a five-year term insurance for the same amount and at the same age. In symbols, we write

$$_{5|}A^{1}_{x:\overline{10}|} = A^{1}_{x:\overline{15}|} - A^{1}_{x:\overline{5}|}.$$

In general, a k-year deferred, n-year term policy would have a net single premium equal to

$$_{k|}A^{1}_{x:\overline{n}|} = A^{1}_{x:\overline{n+k}|} - A^{1}_{x:\overline{k}|} = \frac{M_x - M_{x+n+k}}{D_x} - \frac{M_x - M_{x+k}}{D_x}$$

$$_{k|}A^{1}_{x:\overline{n}|} = \frac{M_{x+k} - M_{x+n+k}}{D_x}. \qquad \text{XLVII}$$

Similarly, it is readily seen that a k-year deferred whole life insurance for \$1 has a single premium equal to that of a whole life insurance for \$1, with protection beginning at once, minus a k-year term insurance for the same amount. In symbols,

$$_{k|}A_x = A_x - A^{1}_{x:\overline{k}|} = \frac{M_x}{D_x} - \frac{M_x - M_{x+k}}{D_x},$$

$$_{k|}A_x = \frac{M_{x+k}}{D_x}. \qquad \text{XLVIII}$$

In the same manner, an n-year endowment insurance of \$1 deferred for k years has a single premium equal to the difference between the net single premiums for an $(n+k)$-year endowment insurance and a k-year term insurance, both for the same amount. In symbols,

$$
\begin{aligned}
{}_{k|}A_{x:\overline{n}|} &= A_{x:\overline{n+k}|} - A^{1}_{x:\overline{k}|} \\
&= \frac{M_x - M_{x+n+k} + D_{x+n+k}}{D_x} - \frac{M_x - M_{x+k}}{D_x} \\
&= \frac{M_{x+k} - M_{x+n+k} + D_{x+n+k}}{D_x}. \qquad \text{XLIX}
\end{aligned}
$$

Formulas for annual premiums are readily obtained from those for single premiums by dividing the corresponding single premium by the present value of an annuity due at the proper age and for the proper number of payments.

Illustrative Example

A certain life insurance policy issued at age 30 is to be paid for by means of 20 annual premiums. In event of the death of the insured within the first 10 years, the death benefit is \$1000; thereafter, it is \$2000. Find the net annual premium.

Let P denote the net annual premium. Upon equating the present value of the net premiums to the present value of the benefits, we have

$$
\begin{aligned}
P\,\ddot{a}_{30:\overline{20}|} &= \$1000\ A^{1}_{30:\overline{10}|} + \$2000\ {}_{10|}A_{30} \\
&= \$1000 \cdot \frac{M_{30} - M_{40}}{D_{30}} + \$2000 \cdot \frac{M_{30+10}}{D_{30}};
\end{aligned}
$$

Then,

$$
\begin{aligned}
P &= \$1000 \cdot \frac{M_{30} + M_{40}}{N_{30} - N_{50}} \\
&= \$49.22.
\end{aligned}
$$

General insurance formula

A single unifying formula may be developed for insurance benefits or single premiums for whole life and term insurances similar to that developed for the value of annuities. Consider the following table of single premiums for whole life and term policies already derived:

Policy	Age at Date of Present Value	Age When Protection Begins	Term in Years	Age at End of Term	Single Premium
Whole life	x	x	life	$\omega+1$	$\frac{M_x}{D_x}$
k-year deferred whole life	x	$x+k$	life	$\omega+1$	$\frac{M_{x+k}}{D_x}$
n-year term	x	x	n	$x+n$	$\frac{M_x - M_{x+n}}{D_x}$
k-year deferred n-year term	x	$x+k$	n	$x+k+n$	$\frac{M_{x+k} - M_{x+k+n}}{D_x}$

From this table, it is apparent that a general formula for the single premium for any of the benefits listed may be written as

$$\frac{M_e - M_f}{D_g} \qquad \text{L}$$

where e is the age at the beginning of the term;
f is the age at the end of the term, or $f = e + n$;
and g is the age at which we wish to value the insurance—the age at issue in all the cases considered heretofore.

We note that, in the case of whole life policies, since coverage extends through the entire table, $f = \omega + 1$ and, since $M_{\omega+1} = 0$, the second term in the numerator vanishes.

This formula does not cover endowment insurance, but this is not a serious matter, inasmuch as the formula for the single premium of any term policy, expressed in commutation symbols, can be converted into that for an endowment insurance policy for the same term and amount and at the same age merely by the addition of the term, D_f, to the numerator.

Given the formula for the single premium for an insurance benefit, we can find the annual premium payable for any specified number of years by dividing the single premium by a life annuity of the appropriate type and number of payments.

Accumulated cost of insurance

A term insurance policy could, theoretically, be paid for by the survivors at the expiration of the term, and we can compute the value of the single premium to be paid at that time. It is obvious that no such policy would be issued, but the notion of this *accumulated cost of insurance* is of value in the subsequent study of reserves. Let us compute this net single premium payable at the end of the term.

As before, we assume that l_x persons take out an n-year term insurance policy for \$1, but this time only the survivors of the n-year period will pay the cost.

At the end of the n-year period, the company will receive $l_{x+n} \cdot {}_nk_x$ dollars, where ${}_nk_x$ represents the accumulated cost of insurance.

Meanwhile, the company will have paid out d_x dollars in death claims at the end of the first year, d_{x+1} at the end of the second, and so forth, until it paid out d_{x+n-1} at the end of the nth year.

The first set of death claims will have accumulated to $(1.03)^{n-1}d_x$ by the end of the period; the second set to $(1.03)^{n-2}d_{x+1}$; etc.

Equating the value of company payments, accumulated to the end of the term, to the value of the company's receipts at that time, yields

$$l_{x+n} \cdot {}_nk_x = (1.03)^{n-1}d_x + (1.03)^{n-2}d_{x+1} + \ldots + d_{x+n-1}.$$

Multiply both sides by v^{x+n}, substitute commutation symbols, and then divide both sides by D_{x+n}. This yields

$${}_nk_x = \frac{C_x + C_{x+1} + \ldots + C_{x+n-1}}{D_{x+n}} = \frac{M_x - M_{x+n}}{D_{x+n}}, \qquad \text{LI}$$

the latter being a simpler expression.

This formula also is a special case of the general insurance formula,
where $e=x$, the age when benefits begin,
$f=x+n$, the age at the end of the term of protection,
and $g=x+n$, the age at which the value is to be determined—in this case, at the end of the term.

Relationship between net single premiums and annuities

A useful relationship between net single insurance premiums and life annuities may now be developed. We begin with the definition of d_x:

$$d_x = l_x - l_{x+1},$$
$$v^{x+1} d_x = v^{x+1} l_x - v^{x+1} l_{x+1} = v(v^x l_x) - v^{x+1} l_{x+1}.$$

Then, from the definitions of the commutation symbols,

$$C_x = vD_x - D_{x+1}.$$

This is a general relationship, valid for all values of x from the initial one up to the end of the table. If we sum each side of all such equations from any given value of x up to the end of the table, we have

$$C_x + C_{x+1} + \dots + C_\omega = v(D_x + D_{x+1} + \dots + D_\omega) - (D_{x+1} + D_{x+2} + \dots + D_\omega),$$

or

$$M_x = vN_x - N_{x+1}.$$

Now, dividing both sides by D_x,

$$\frac{M_x}{D_x} = \frac{vN_x}{D_x} - \frac{N_{x+1}}{D_x},$$

or

$$A_x = v\text{a}_x - a_x. \qquad \text{LII}$$

This last equation may be expressed in another form by defining d, the *rate of discount*, as equal to iv. That is,

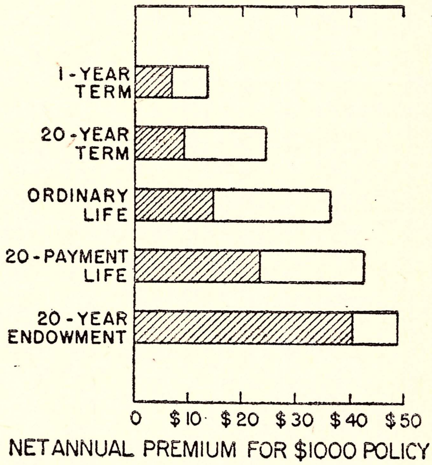
$$d = iv = \frac{i}{1+i} = \frac{1+i-1}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i}$$
$$= 1 - v,$$

so that $v = 1 + d$.

Also, since $\text{a}_x = 1 + a_x$, we may substitute $\text{a}_x - 1$ for a_x. We then have

$$A_x = (1+d)\text{a}_x - (\text{a}_x - 1) = 1 - d \cdot \text{a}_x. \qquad \text{LIII}$$

A similar relationship between the single premium for an endowment insurance and a temporary life annuity may likewise be derived.



Comparison of Net Annual Premiums at Ages 20 (shaded) and 50 (unshaded) for Various Types of Policies

Fig. 6

The following set of miscellaneous problems is intended to test the reader's mastery of the entire section on life insurance net premiums, including the accumulated cost of insurance:

TEST YOUR KNOWLEDGE OF LIFE INSURANCE PREMIUMS

49 What amount of ordinary life insurance can be purchased by a man of 18 for $20 per year?

50 A whole life insurance policy for $1000 taken at age 25 provides that the insurance of the first year is term insurance, and thereafter the

insurance is an ordinary life policy. What is the net premium for (a) the first year, and (b) any subsequent year?

51 Find the single premium for a $1000, thirty-year endowment insurance issued at age 23.

52 A man aged 32 agrees to pay $35 per year for a ten-year term insurance policy. What is the face amount of the policy?

53 Find the accumulated cost of insurance at the end of 5 years on a $1000 policy issued at age 44.

54 A certain policy, often called a "modified life" policy, provides a death benefit of $1000 and extends through the whole of life. Each of the first 5 annual premiums is exactly one-half the size of each of the remainder of the premiums, from the sixth year onward. Compute the size of both types of premiums for such a policy issued to a man aged 25.

55 An endowment insurance policy at age 33 has a face amount sufficient to provide the insured with a life income of $600, first payment at the maturity date of the endowment at age 65. If annual premiums are payable up to age 60 (*i.e.*, there are 27 premiums), find their size.

56 Derive the relationship,

$$A_{x:\overline{n|}} = v\ddot{a}_{x:\overline{n|}} - a_{x:\overline{n-1|}}.$$

VARYING INSURANCES AND ANNUITIES

Heretofore we have considered only annuities which consisted of payments of uniform size and insurances under which the size of the death benefit remained constant throughout the term of the policy. There are policies written under which the insurance increases (or decreases) at a constant rate and occasionally one finds a policy under which premiums increase (or decrease) regularly, at least for a time, so that it is of more than academic interest to study varying annuities and insurances.

Increasing and decreasing life annuities

Let us consider a whole life annuity due issued to a man aged x, which provides that the first payment (at age x) shall be $1, the second payment shall be $2, the third payment shall be $3, and so on. We shall say that such an annuity has a step of $1. Looking at each individual payment as a pure endowment, the present value of the entire annuity, $(I\ddot{a})_x$, will equal the sum of the present values of these pure endowments, and we then have

$$(I\ddot{a})_x = \frac{D_x + 2D_{x+1} + 3D_{x+2} + \ . \ . \ . \ + (\omega - x + 1)D_\omega}{D_x}.$$

To simplify this numerator, note that

$$\begin{aligned}
N_x &= D_x + D_{x+1} + D_{x+2} + D_{x+3} + \ . \ . \ . \ + D_\omega; \\
N_{x+1} &= \quad\ D_{x+1} + D_{x+2} + D_{x+3} + \ . \ . \ . \ + D_\omega; \\
N_{x+2} &= \qquad\qquad D_{x+2} + D_{x+3} + \ . \ . \ . \ + D_\omega; \\
&\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\
N_\omega &= \qquad\qquad\qquad\qquad\qquad\qquad\quad D_\omega.
\end{aligned}$$

If we add these equations, the sum of the right members will equal the

numerator in the above fraction. Hence, it may be replaced by the left member—that is, by $N_x+N_{x+1}+N_{x+2}+ \ldots +N_\omega$, so that

$$(Ia)_x=\frac{N_x+N_{x+1}+N_{x+2}+ \ldots +N_\omega}{D_x}.$$

The form of this expression may be further simplified by the introduction of a new commutation symbol, S_x, which is defined by

$$S_x=N_x+N_{x+1}+N_{x+2}+ \ldots +N_\omega.$$

Then we have
$$(Ia)_x=\frac{S_x}{D_x}. \qquad \text{LIV}$$

In like manner, it may be shown that the present value of an increasing whole life annuity immediate is

$$(Ia)_x=\frac{S_{x+1}}{D_x}. \qquad \text{LV}$$

Values of S_x appear in Table LXXXV.

It is fairly easy to see that the present value of an n-year temporary increasing life annuity due, with a step of \$1, to a man aged x would be

$$(I\ddot{a})_{x:\overline{n}|}=\frac{D_x+2D_{x+1}+3D_{x+2}+ \ldots +nD_{x+n-1}}{D_x}.$$

To simplify this, note that

$$\begin{aligned}
N_x-N_{x+n}&=D_x+D_{x+1}+D_{x+2}+ \ldots +D_{x+n-1};\\
N_{x+1}-N_{x+n}&=\phantom{D_x+{}}D_{x+1}+D_{x+2}+ \ldots +D_{x+n-1};\\
N_{x+2}-N_{x+n}&=\phantom{D_x+D_{x+1}+{}}D_{x+2}+ \ldots +D_{x+n-1};\\
&\vdots\\
N_{x+n-1}-N_{x+n}&=\phantom{D_x+D_{x+1}+D_{x+2}+ \ldots +{}}D_{x+n-1}.
\end{aligned}$$

Summing these n equations, we have

$$N_x+N_{x+1}+N_{x+2}+ \ldots +N_{x+n-1}-nN_{x+n}=D_x+2D_{x+1}+3D_{x+2}+ \ldots +nD_{x+n-1}.$$

The right member is the numerator of the expression for $(I\ddot{a})_{x:\overline{n}|}$ above. We may then write

$$(I\ddot{a})_{x:\overline{n}|}=\frac{N_x+N_{x+1}+N_{x+2}+ \ldots +N_{x+n-1}-nN_{x+n}}{D_x}.$$

From the definition of S_x, it is evident that

$$N_x+N_{x+1}+N_{x+2}+ \ldots +N_{x+n-1}=S_x-S_{x+n},$$

so we reach the simplified form,

$$(I\ddot{a})_{x:\overline{n}|}=\frac{S_x-S_{x+n}-n\cdot N_{x+n}}{D_x}. \qquad \text{LVI}$$

Similarly, for the n-year temporary increasing life annuity immedate, we have

$$(Ia)_{x:\overline{n}|}=\frac{S_{x+1}-S_{x+n}-nN_{x+n+1}}{D_x}. \qquad \text{LVII}$$

An n-year temporary *decreasing* life annuity due with a step of \$1, issued to a man aged x, pays n dollars at age x, $n-1$ dollars at age $x+1$, $n-2$ dollars at age $x+2$, and so on until the last payment of

\$1 is made at age $x+n-1$. The present value of this annuity may be derived in a manner analagous to that used above for increasing annuities. An alternative method consists in considering such a decreasing annuity as the difference between a level n-year temporary annuity due for $n+1$ dollars and an n-year temporary increasing annuity due with a step of \$1. In this latter case, we have

$$(Da)_{x:\overline{n}|}=(n+1)a_{x:\overline{n}|}-(Ia)_{x:\overline{n}|}$$

$$=(n+1)\cdot\frac{N_x-N_{x+n}}{D_x}-\frac{S_x-S_{x+n}-nN_{x+n}}{D_x}$$

$$=\frac{nN_x-(S_x-N_x)+(S_{x+n}-N_{x+n})-nN_{x+n}+nN_{x+n}}{D_x}$$

$$=\frac{nN_x-S_{x+1}+S_{x+n+1}}{D_x}. \qquad \text{LVIII}$$

Various combinations of level and increasing or decreasing annuities may be devised, and their present values obtained, either directly or by analysis into their component parts.

Illustrative Example A

Find the present value of an annuity to a man aged 31, which pays \$250 at the end of the first year, \$300 at the end of the second, and so on, the payments increasing by \$50 each year until \$500 is paid at the end of the sixth year, after which payments cease.

This annuity can evidently be considered as the sum of a level annuity of \$200 per year plus an increasing annuity with a step of \$50. The value will be

$$\$200\,a_{31:\overline{6}|}+\$50(Ia)_{31:\overline{6}|}=\$200\cdot\frac{N_{32}-N_{38}}{D_{31}}+\$50\cdot\frac{S_{32}-S_{38}-6N_{38}}{D_{31}}$$

$$=\frac{\$200(673{,}396-495{,}187)+\$50(10{,}774{,}650-7{,}199{,}753-2{,}971{,}122)}{33{,}887.3}$$

$$=\$1942.63.$$

Illustrative Example B

Compute the present value of an annuity to a man aged 40 which pays \$10 at the beginning of the first year, then \$20, \$30, etc. until \$100 per year is reached, after which payments remain at this level for life.

This annuity can be considered as a 10-year increasing annuity due with a step of \$10, plus a 10-year deferred level whole life annuity due of \$100. The present value is

$$\$10\,(Ia)_{40:\overline{10}|}+\$100\;{}_{10|}a_{40}$$

$$=\$10\cdot\frac{S_{40}-S_{50}-10N_{50}}{D_{40}}+\$100\cdot\frac{N_{50}}{D_{40}}=\$10\cdot\frac{S_{40}-S_{50}}{D_{40}}$$

$$=\$10\cdot\frac{6{,}235{,}271-2{,}763{,}949}{23{,}943{,}9}=\$1449.77.$$

Illustrative Example C

Find the present value of an annuity to a man aged 53 which pays \$1500 at the beginning of the first year, \$1250 the second, \$1000 the third, and decreasing thus to zero.

This decreasing annuity will run for 6 years and has a step of \$250. Its present value is

$$\$250(Da)_{53:\overline{6|}} = \$250 \cdot \frac{6N_{53} - S_{54} + S_{60}}{D_{53}}$$

$$= \$250 \cdot \frac{1{,}184{,}406 - 1{,}884{,}172 + 971{,}126}{13{,}943.9} = \$4865.21.$$

TEST YOUR KNOWLEDGE OF INCREASING AND DECREASING ANNUITIES

57 What should a man aged 26 pay for a whole life annuity paying \$10 at the end of the first year and increasing by \$10 per year thereafter for life?

58 Find the present value of a life annuity to a man now aged 61, beginning with an initial payment at once of \$500 and decreasing by steps of \$25 per year to zero?

59 What is the present value of a life annuity to a man aged 47 beginning at the end of the year with a payment of \$25 and increasing by \$5 per year until a payment size of \$100 is reached, after which the annuity payments remain constant throughout life?

60 A man now aged 36 has a temporary life annuity with successive payments of \$500, \$450, \$400, \$350, \$300, and \$250, the first payment to be made immediately. Compute the present value of this annuity.

61 Find the present value to a man aged 48 of a life annuity which pays \$500, \$600, \$700, \$800, \$900, \$1000, \$900, \$800, \$700, \$600, and \$500 at the beginning of each of the successive years.

Increasing and decreasing insurances

Probably the most common applications of varying insurances arise in the following cases:

a Most states limit the amount of death benefit payable on children, so that it is usual for children's life insurance policies to provide for a benefit which increases in the early years to the ultimate level which is maintained thereafter.

b Some companies issue policies offering protection which tapers off in the later years of life. The theory is that a man's estate increases with his age and his responsibilities to dependents decrease with age; hence, his need for life insurance protection tends to decrease.

c Deferred annuities paid for by annual premiums are sometimes sold with the guarantee that all premiums paid will be returned if death occurs before the prospective annuitant has entered upon his annuity payments. Sometimes insurance policies are sold with the "feature" that all premiums paid will be returned upon the death of the insured in addition to the payment of the face of the policy. Policies of this sort are known as *return of premium policies.* They will be discussed in detail on pages 938 and 939.

Let us consider an *increasing* whole life insurance under which the death benefit is \$1 the first year, \$2 during the second year, \$3 the third, and so on for life. The single premium for such a policy issued at age x is

$$(IA)_x=\frac{C_x+2C_{x+1}+3C_{x+2}+\ .\ .\ .\ +(\omega-x+1)C_\omega}{D_x},$$

which, as in the analagous case of the increasing whole life annuity can be written more simply as

$$(IA)_x=\frac{M_x+M_{x+1}+M_{x+2}+\ .\ .\ .\ M_\omega}{D_x}.$$

By the introduction of another commutation symbol, R_x, defined by the equation,

$$R_x=M_x+M_{x+1}+M_{x+2}+\ .\ .\ .\ +M_\omega,$$

the expression for $(IA)_x$ may be put in the very simple form,

$$(IA)_x=\frac{R_x}{D_x}. \qquad \text{LIX}$$

In a similar fashion, the single premium for the n-year increasing term insurance is found to be

$$(IA)^{1}_{x:\overline{n|}}=\frac{R_x-R_{x+n}-nM_{x+n}}{D_x}. \qquad \text{LX}$$

An n-year *decreasing* term insurance with a step of \$1 offers a death benefit of n dollars if the insured dies during the first year, $n-1$ dollars if he dies during the second year, and so on until the benefit is \$1 for death during the nth year and zero thereafter. As in the analagous case of the decreasing temporary life annuity, the decreasing insurance may be considered as the difference between a level insurance and an increasing insurance. Then the single premium for an n-year decreasing term insurance with a step of \$1, issued to a man aged x, is

$$\begin{aligned}(DA)^{1}_{x:\overline{n|}}&=(n+1)A^{1}_{x:\overline{n|}}-(IA)^{1}_{x:\overline{n|}}\\ &=(n+1)\cdot\frac{M_x-M_{x+n}}{D_x}-\frac{R_x-R_{x+n}-nM_{x+n}}{D_x}\\ &=\frac{nM_x-(R_x-M_x)+(R_{x+n}-M_{x+n})-nM_{x+n}+nM_{x+n}}{D_x}\\ &=\frac{nM_x-R_{x+1}+R_{x+n+1}}{D_x}. \qquad \text{LXI}\end{aligned}$$

The corresponding formulas for annual premiums may be derived by dividing the single premium by the present value of the appropriate life annuity due.

Values of R_x may be found in Table LXXXV.

Illustrative Example A

Find the single premium of an increasing whole life insurance to a man aged 49 if the protection offered is \$100 the first year, \$200 the second year, and so on.

The single premium is

$$\$100(IA)_{49}=\$100\cdot\frac{R_{49}}{D_{49}}=\$100\cdot\frac{171{,}705}{16{,}618.3}=\$1033.23.$$

Illustrative Example B

Compute the value of the single premium for a decreasing life insurance to a man aged 38, under the terms of which the protection is \$1000 the first year, \$950 the second, and so on until it reaches zero.

The single premium is

$$\$50(DA)^{1}_{38:\overline{20}|}=\$50\cdot\frac{20M_{38}-R_{39}+R_{59}}{D_{38}}$$
$$=\$50\cdot\frac{229{,}374.0-274{,}017+91{,}004.4}{25{,}891.6}$$
$$=\$89.53.$$

Illustrative Example C

Find the single premium for a child's policy under which the protection is \$100 for the first year and increases by \$100 each year until \$1000 is reached and remains at that level for life, if the policy is issued to a child aged 10.

This problem can be solved in more than one way. We shall consider two different analyses.

First Solution

The protection may be considered as that of a 10-year increasing term insurance with a step of \$100 plus a 10-year deferred whole life insurance for \$1000. Then the single premium will consist of the sum,

$$\$100(IA)^{1}_{10:\overline{10}|}+\$1000\ {}_{10|}A_{10}$$
$$=\$100\cdot\frac{R_{10}-R_{20}-10M_{20}}{D_{10}}+\$1000\cdot\frac{M_{20}}{D_{10}}$$
$$=\$100\cdot\frac{R_{10}-R_{20}}{D_{10}}=\$100\cdot\frac{734{,}216-550{,}028}{74{,}409.4}$$
$$=\$260.97.$$

Second Solution

The protection may be considered as that of a \$1000 whole life insurance minus a 9-year decreasing term insurance with a step of \$100. Under this interpretation, the single premium would consist of the difference,

$$\$1000A_{10}-\$100(DA)^{1}_{10:\overline{9}|}$$
$$=\$1000\cdot\frac{M_{10}}{D_{10}}-\$100\cdot\frac{9M_{10}-R_{11}+R_{20}}{D_{10}}$$
$$=\$100\cdot\frac{10M_{10}-9M_{10}+R_{11}-R_{20}}{D_{10}}=\$100\cdot\frac{M_{10}+R_{11}-R_{20}}{D_{10}}.$$

Since $M_{10}+R_{11}=R_{10}$, the above solution becomes

$$\$100\cdot\frac{R_{10}-R_{20}}{D_{10}}.$$

This is identical with the first solution.

TEST YOUR KNOWLEDGE OF INCREASING AND DECREASING INSURANCES

62 Find the single premium for an insurance issued at age 42, providing for a death benefit of $1000 the first year, $950 the second year, and decreasing by $50 per year until it reaches zero.

63 Compute the annual premium payable for 20 years of a whole life policy issued at age 25 and providing for a death benefit of $1000 the first year, $1100 the second year, $1200 the third, and so on for life.

64 A certain policy issued at age 29 provides for $3000 of insurance up to age 51, then $2900 for death between ages 51 and 52, $2800 for death between ages 52 and 53, and so on, decreasing by $100 per year until the amount of insurance reaches $1000 at age 70, after which it remains constant. Find (a) the net single premium and (b) the net annual premium payable from age 29 to age 64 inclusive.

65 A child's endowment policy issued at age 10 provides for a death benefit of $500 the first year, $600 the second, and so on, increasing by $100 per year until a maximum of $1000 is reached. The policy matures at age 21 as an endowment for $1000. Find the size of the net annual premium.

66 A 20-year endowment insurance policy for $1000 is paid for by annual premiums which vary as follows: the second premium is double the first, the third is three times the first, the fourth is four times the first, and the fifth and all subsequent premiums throughout the contract are level at five times the initial premium. Find the size of these premiums for such a policy issued to a man aged 30.

Return of premium policies

It is fairly common for an insurance company to make a deferred annuity contract more attractive to a prospective purchaser by offering him some return in the event of his death prior to the date at which annuity payments commence. One such contract promises the return of all the annuitant's premiums paid, without interest. Let us consider an example of this sort.

Illustrative Example A

Suppose that a deferred annuity for $1000 per year, first payment to be made at age 65, is sold to a man aged 25, who agrees to pay equal annual premiums up to, but not including, age 65. It is agreed that the total of the net premiums paid will be returned, without interest, in the event of his death prior to age 65.

To find the size of the net annual premiums, P, equate the present value of future premiums to the present value of all future benefits. The benefits are of two kinds: (a) a deferred whole life annuity and (b) a 40-year increasing term insurance with a step of $\$P$. The equation then is

$$P a_{25:\overline{40|}} = \$1000\ {}_{40|}a_{25} + P(IA)^{1}_{25:\overline{40|}},$$

or

$$P \cdot \frac{N_{25}-N_{65}}{D_{25}} = \$1000 \cdot \frac{N_{65}}{D_{25}} + P \cdot \frac{R_{25}-R_{65}-40M_{65}}{D_{25}}.$$

Solving for the net annual premium, P, we have

$$P=\$1000 \cdot \frac{N_{65}}{N_{25}-N_{65}-R_{25}+R_{65}+40M_{65}}.$$

This expression can be readily evaluated by substituting the values of the commutation symbols from the tables.

In practice, it is usually the gross premium, G, which is returned at death. If we know the relationship between net and gross premiums, equating the present value of future net premiums to the present value of future benefits, including the return of the gross premiums paid, will enable us to find the net, and hence the gross premiums for the policy. Since the relationship between net and gross premiums is often rather complicated, we shall not go further into this problem.

It sometimes happens that a life insurance contract will provide for the return of all premiums paid, without interest, in addition to the face amount of the policy as a death benefit. Let us consider such an example.

Illustrative Example B

A life insurance policy, issued at age 33 for a face amount of \$1000, provides for the payment, at death, of this face amount plus the sum of all annual premiums paid up to the time of death. Find the net annual premium for this policy which continues for the whole of life.

Again, we equate the present value of all future net premiums to the present value of all future benefits. Then, using P as the net premium to be found, we have

$$Pa_{33}=\$1000A_{33}+P(IA)_{33}$$

$$P \cdot \frac{N_{33}}{D_{33}}=\$1000 \cdot \frac{M_{33}}{D_{33}}+P \cdot \frac{R_{33}}{D_{33}}.$$

Solving for P, we have

$$P=\frac{\$1000M_{33}}{N_{33}-R_{33}}.$$

It is evident that further combinations of return of premium annuities and insurances can be devised.

TEST YOUR KNOWLEDGE OF RETURN OF PREMIUM POLICIES

67 Complete the solution for the net premium in Illustrative Examples A and B.

68 Compute the net annual premium, payable for 20 years, for an annuity of \$1000 per year beginning at age 55, if the annuity is issued at age 25 and provides, further, for the return of all net premiums paid if the annuitant dies before reaching age 55.

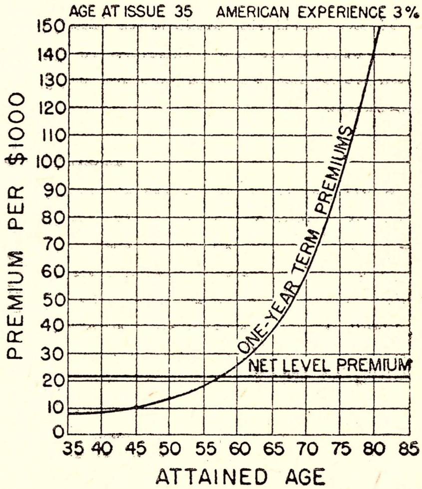
69 A special 20-year endowment policy issued at age 40 provides in the event of the death of the insured during the 20-year period for the payment of \$1000 plus all net premiums previously paid. If the insured survives the twenty years, the policy matures for \$1000 only. Find the net annual premium.

· LIFE INSURANCE RESERVES AND NON-FORFEITURE BENEFITS ·

By C. J. Nesbitt, Ph.D.

TWO of the major mathematical problems of an insurance organization are the determination of premiums and the evaluation of reserves. The first problem has been dealt with on a theoretical basis in the preceding article, where formulas are given for net level premiums. During the early years of an insurance contract, the level premium is more than sufficient to cover the theoretical insurance cost, an excess each year being available for accumulation. These accumulated excess premiums form the reserve.

For example, the net level premium, on the American Experience 3% basis, for a $1000 ordinary life policy issued at age 35 is $21.08. The net premium for one-year term insurance, on the same mortality and interest bases, begins at $8.69 at age 35, and increases until at age 58 it overtakes the ordinary life premium, and continues to increase still more rapidly so that at age 85, for instance, it is $228.69 (see Fig. 7). The net premium for one-year term insurance at a given age represents the theoretical cost of the insurance benefit for the year that will follow. Thus, from age 35 up to age 57, the ordinary life net premium is in excess of the theoretical insurance cost, and a reserve fund may be accumulated. In fact, a reserve *must* be accumulated, for, from age 58 onward, the net level premium is less than the theoretical insurance cost, and the reserve fund and its interest earnings must be available to make up the deficiency. The advantage of the net level premiums is thereby indicated; by paying more in the early policy years than the theoretical cost, the policyholder is able to have his insurance in the later policy years for much less than the rapidly rising cost.



Ordinary Life Net Level Premium Compared with One-Year Term Premiums

Fig. 7

From there, we shall go on to the consideration of modified reserves and of non-forfeiture benefits.

NET LEVEL PREMIUM RESERVES

We shall consider here a group of $l_{35} = 81{,}822$ persons, each aged exactly 35, whose mortality is assumed to follow exactly the American Experience Table. Let us suppose that each member of the group takes out with the Nonsuch Insurance Company

a $1000 five-year term insurance policy. We shall ignore at this stage the factor of expense, and assume that the premium charged is the net level premium,

$$\$1000P^{1}_{35:\overline{5}|}=\$8.971.$$

The Nonsuch then receives, at date of issue of the policies to the group, premiums totalling

$$\$l_{35}\cdot 1000P^{1}_{35:\overline{5}|}=\$734{,}025.$$

During the first policy year, these premiums will earn, at 3%, interest amounting to (0.03) (734,025) = $22,021 to the nearest dollar, but d_{35} members will die during the year, and at the end of the year claims amounting to 1000d_{35}$(= $732,000) will be paid by the Nonsuch to the proper beneficiaries. The Nonsuch will then have a fund for the group amounting to

$$\$734{,}025+\$22{,}021-\$732{,}000=\$24{,}046,$$

or, since now there are only l_{36} (= 81,090) members surviving, an amount of

$$\frac{\$24{,}046}{81{,}090}=\$0.30$$

per survivor.

At the beginning of the second policy year, the l_{36} survivors of the group will pay premiums totalling $\$l_{36}\cdot 1000P^{1}_{35:\overline{5}|}(=\$727{,}458)$.

This, together with the $24,046 left at the end of the first year's operations, will make $751,504 available for accumulation during the second year; under 3% interest, this latter will earn $22,545 by the end of the second year.

The second-year claims will amount to 1000d_{36}$, so that the Nonsuch will have on hand at the end a total fund for the group of

$$\$751{,}504+\$22{,}545-\$737{,}000=\$37{,}049,$$

or, dividing by l_{37} (= 80,353), $0.46 for each of the l_{37} survivors. Then, at the beginning of the third year, there will be

$$\$37{,}049+\$l_{37}\cdot 1000P^{1}_{35:\overline{5}|}=\$757{,}896$$

available for accumulation.

We proceed to compute, as in the first and second policy years, the group fund and the individual survivor's share at the end of the third year, and continue the process for the fourth and fifth policy years. The results are tabulated in Table A.

The amount per survivor, at the end of t years, of the group fund that has accumulated by that time, is called the tth reserve or, more exactly, the tth *terminal reserve*, to denote that it is the reserve at the *end* of t years.

Note that the total outgo of $3,716,000 for claims is offset almost exactly by the two incomes—namely, $3,603,991 of premium income, and $111,904 of interest income. If the tabular values used to compute $P^{1}_{35:\overline{5}|}$ had been given to more figures, and the computations correspondingly extended, then these totals would agree exactly. Also, observe, that at the end of the fifth year, when the term insurance expires, the reserve per policy is zero.

In Table B, we present the reserve accumulation for a five-year

TABLE A

RESERVE ACCUMULATION FOR FIVE-YEAR TERM INSURANCE ISSUED AT AGE 35

(1) Policy Year	(2) Survivors of Group at Beginning of Year	(3) Premiums Paid by Group	(4) Total Fund at Beginning of Year	(5) Interest Earned During Year	(6) Claims Payable at End of Year	(7) Total Fund at End of Year (4)+(5)−(6)	(8) Reserve at End of Year
1	81,822	$734,025	$734,025	$22,021	$732,000	$24,046	$0.30
2	81,090	727,458	751,504	22,545	737,000	37,049	0.46
3	80,353	720,847	757,896	22,737	742,000	38,633	0.49
4	79,611	714,190	752,823	22,585	749,000	26,408	0.33
5	78,862	707,471	733,879	22,016	756,000	−105	0.00
Totals		$3,603,991		$111,904	$3,716,000		

endowment insurance, assuming now that each of our l_{35} group members takes out a $1000 five-year endowment policy, the premium for which is

$$\$1000P_{35:\overline{5}|}\ (=\$186.640).$$

The reserve accumulation proceeds in the same manner as for the five-year term insurance, but now the interest earnings are much higher, as larger funds are available for investment. At the end of the fifth year, the policies mature, and for each surviving member there must be on hand the face amount, so that in this case the fifth terminal reserve is $1000.

Note that the total income,

$$\$74{,}980{,}381+\$6{,}841{,}063=\$81{,}821{,}444,$$

from premiums and interest, is closely sufficient to meet the total claims of $81,822,000, consisting of $3,716,000 for death claims, and $\$1000l_{40}(=\$78{,}106{,}000)$ of maturity claims.

In Table C is given the reserve accumulation, during the first five years, for ordinary life insurance. Here each member of the group takes out a $1000 ordinary life policy, for which he pays an annual premium of $\$1000P_{35}(=\$21.081)$.

Here the total income of $8,469,039+$570,567 (=$9,039,606) from premiums and interest, is almost exactly sufficient to pay the total death claims of $3,716,000 and, in addition, to have at the end

TABLE B

RESERVE ACCUMULATION FOR FIVE-YEAR ENDOWMENT INSURANCE ISSUED AT AGE 35

(1) Policy Year	(2) Survivors of Group at Beginning of Year	(3) Premiums Paid by Group	(4) Total Fund at Beginning of Year	(5) Interest Earned During Year	(6) Claims Payable at End of Year	(7) Total Fund at End of Year (4)+(5)−(6)	(8) Reserve at End of Year
1	81,822	$15,271,258	$15,271,258	$458,138	$732,000	$14,997,396	$184.95
2	81,090	15,134,638	30,132,034	903,961	737,000	30,298,995	377.07
3	80,353	14,997,084	45,296,079	1,358,882	742,000	45,912,961	576.72
4	79,611	14,858,597	60,771,558	1,823,147	749,000	61,845,705	784.23
5	78,862	14,718,804	76,564,509	2,296,935	756,000	78,105,444	999.99
Totals		$74,980,381		$6,841,063	$3,716,000		

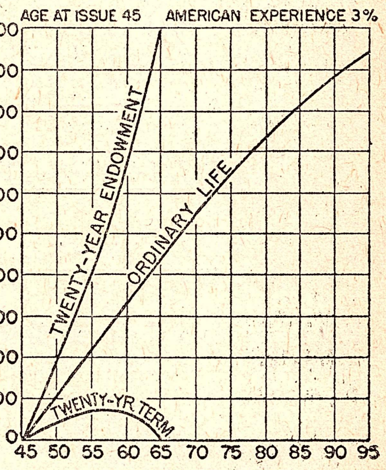
TABLE C

RESERVE ACCUMULATION FOR ORDINARY LIFE INSURANCE ISSUED AT AGE 35

(1) Policy Year	(2) Survivors of Group at Beginning of Year	(3) Premiums Paid by Group	(4) Total Fund at Beginning of Year	(5) Interest Earned During Year	(6) Claims Payable at End of Year	(7) Total Fund at End of Year (4)+(5)−(6)	(8) Reserve at End of Year
1	81,822	$1,724,890	$1,724,890	$51,747	$732,000	$1,044,637	$12.88
2	81,090	1,709,458	2,754,095	82,623	737,000	2,099,718	26.13
3	80,353	1,693,922	3,793,640	113,809	742,000	3,165,449	39.76
4	79,611	1,678,279	4,843,728	145,312	749,000	4,240,040	53.77
5	78,862	1,662,490	5,902,530	177,076	756,000	5,323,606	68.16
Totals		$8,469,039		$570,567	$3,716,000		

of the fifth year a reserve of $68.16 for each of the l_{40} $(=78{,}106)$ survivors.

As further illustration of reserve accumulations, the graphs of the terminal reserves for some typical $1000 policies, issued at age 45, are shown in Fig. 8. For convenience, these graphs are shown as smooth curves, although actually the values jump after each premium is paid (see Fig. 9). Note the relative smallness of the reserves on the twenty-year term policy, and note also that the twenty-year endowment reserve reaches $1000 by the time the policy is ready to mature.



Graphs of Terminal Reserves for Typical $1000 Policies

Fig. 8

Retrospective reserve formulas

In the section just preceding, we computed reserves by filling in accumulation schedules. The schedule method, however, is both bulky and lengthy, and we now seek algebraic means for more readily calculating reserve values.

Let us assume that we have a group of l_x persons aged x, each of whom takes out identical insurance policies for the face amount of $1, that the premium for such a policy is P, and that premiums are payable for m years. We shall obtain a formula for the tth terminal reserve on such policies, first for the case that $t<m$.

Disregarding the death claims, we find that the premiums paid by the group will accumulate at the end of t years as follows:

Premiums, totalling l_xP, will be paid at the beginning by the group, and these will accumulate to $l_xP(1+i)^t$ by the end of the tth year.

There will be only l_{x+1} survivors at the beginning of the second year, and their premiums will accumulate to $l_{x+1}P(1+i)^{t-1}$ by the end of the tth year. Proceeding in this manner, we see that, by the end of the tth year, the premiums paid during the first t years will accumulate the total fund of

$$l_xP(1+i)^t+l_{x+1}P(1+i)^{t-1}+\ \ldots\ +l_{x+t-1}P(1+i) \qquad \text{I}$$

if the death claims are not paid out.

At the end of the first year, d_x claims, each for \$1, will be paid. If these d_x dollars were left in the fund, they would accumulate to $d_x(1+i)^{t-1}$ by the end of the tth year. Similarly, the d_{x+1} claims payable at the end of the second year would accumulate to $d_{x+1}(1+i)^{t-2}$, and, proceeding, the total accumulation of the death claims would be

$$d_x(1+i)^{t-1}+d_{x+1}(1+i)^{t-2}+ \ . \ . \ . \ +d_{x+t-1} \qquad \text{II}$$

by the end of t years.

Actually, the death claims will be paid at the end of the year of death, so that the death claims are not left in the fund to accumulate. Then I overstates the accumulation by the amount, II, to which the death claims would accumulate. Hence, the actual accumulated fund at the end of t years will be I−II, or the reserve, ${}_tV$, for each of the l_{x+t} survivors will be $\frac{\text{I}-\text{II}}{l_{x+t}}$, which gives

$${}_tV=P\frac{l_x(1+i)^t+l_{x+1}(1+i)^{t-1}+ \ . \ . \ . \ +l_{x+t-1}(1+i)}{l_{x+t}}$$
$$-\frac{d_x(1+i)^{t-1}+d_{x+1}(1+i)^{t-2}+ \ . \ . \ . \ +d_{x+t-1}}{l_{x+t}}.$$

Recalling from the preceding article formula XXVI for the accumulated value of an annuity, and LI for the accumulated cost of insurance, we see that

$${}_tV=P\cdot\frac{N_x-N_{x+t}}{D_{x+t}}-\frac{M_x-M_{x+t}}{D_{x+t}} \qquad \text{III}$$

or, for convenience, let us use the briefer notation,

$${}_tV=P{}_tu_x-{}_tk_x. \qquad \text{IV}$$

Formulas III and IV state that the tth terminal reserve may be calculated by taking the accumulated value at the end of t years of the annuity of annual net premiums and subtracting the accumulated cost of the insurance benefit during the t years. Observe that the reserve at the end of t years is here calculated by accumulating the income and the outgo of the t years that have preceded the reserve date. For this reason, they are called *retrospective formulas.*

These formulas hold for a policy with death benefit of \$1, and with number of premiums m exceeding t. The reserve for a policy with death benefit of \$$S$ is obtained by merely multiplying the right-hand side of III or IV by S. For the case, $t\geqq m$, the formula required is

$${}_tV=P\cdot\frac{N_x-N_{x+m}}{D_{x+t}}-\frac{M_x-M_{x+t}}{D_{x+t}}, \qquad \text{V}$$

since premiums are payable only between ages x and $x+m$, and by the

general annuity formula, XXV, of the previous article, their accumulated value at the end of t years will be $P \cdot \frac{N_x - N_{x+m}}{D_{x+t}}$.

STANDARD SYMBOLS

We consider here policies of face amount $1, issued at age x. Standard symbols are used to denote the terminal reserves. For the tth terminal reserves on such policies, the symbols are:

$_tV_x$, for a whole life policy;
$_tV^{1}_{x:\overline{n}|}$, for an n-year term insurance;
$_tV_{x:\overline{n}|}$, for an n-year endowment insurance;
$_{t:m}V_x$, for an m-payment whole life policy.

Illustrative Example

Compute the fifth and thirtieth terminal reserves on a $1000 twenty-payment life policy, issued at age 30.

The first step is to find the value of the premium, $\$1000\,_{20}P_{30}$, which is given in Table LXXXVI as $27.186.

Then, from formula III,

$$\begin{aligned}\$1000\;_{5:20}V_{30} &= \$27.186 \cdot \frac{N_{30}-N_{35}}{D_{35}} - \$1000 \cdot \frac{M_{30}-M_{35}}{D_{35}}\\ &= \frac{27.186(163{,}323)-1000(1365.4)}{29078.2}\\ &= \$105.74.\end{aligned}$$

For the thirtieth reserve, we have from formula V,

$$\begin{aligned}\$1000\;_{30:20}V_{30} &= \$27.186 \cdot \frac{N_{30}-N_{50}}{D_{60}} - \$1000 \cdot \frac{M_{30}-M_{60}}{D_{60}}\\ &= \frac{27.186(499{,}328)-1000(7020.67)}{9830.43}\\ &= \$666.71.\end{aligned}$$

TEST YOUR KNOWLEDGE OF RETROSPECTIVE RESERVE FORMULAS

1 Compute the value of $1000\,_3V^{1}_{35:\overline{5}|}$, the third terminal reserve on a $1000 five-year term insurance issued to a man aged 35. Check your answer by Table A.

2 Compute the value of $1000\,_3V_{35:\overline{5}|}$, the third terminal reserve on a $1000 five-year endowment insurance issued to a man aged 35. Check your answer by Table B.

3 Compute the value of $1000\,_3V_{35}$, the third terminal reserve on a $1000 ordinary life insurance issued to a man aged 35. Check your answer by Table C.

4 Compute the values of $\$2500\;_{15:20}V_{30}$ and $\$2500\;_{25:20}V_{30}$, which are the fifteenth and twenty-fifth terminal reserves on a $2500 twenty-payment whole life policy issued to a man aged 30.

Prospective reserve formulas

Let us consider from a new viewpoint the reserve-fund problems of Nonsuch Insurance Company, which has issued \$1000 ordinary life policies to a group of l_{35} persons aged 35. In Table C, we obtained the fifth terminal reserve on such a policy by means of an accumulation schedule. At the end of the fifth year, the company looks towards the future and realizes that it must provide \$1000 insurance for the remainder of life for each of the survivors, now aged 40. The present value of each of these insurances is $\$1000A_{40}$. The company has two sources from which to make provision for these insurances, one being the future premiums, the other being the reserve already on hand. For a single policy, the present value of the future premiums is $\$1000P_{35}\,a_{40}$. Let us denote the fifth reserve by its standard symbol, $1000\;{}_5V_{35}$. If these two sources are to provide exactly for the insurance benefit, then,

$$1000\;{}_5V_{35}+1000P_{35}\,a_{40}=1000A_{40}$$

or

$$\begin{aligned}1000\;{}_5V_{35}&=1000A_{40}-1000P_{35}\,a_{40}\\&=459.423-21.081(18.5598)\\&=68.16,\end{aligned}$$

which agrees with our former calculation.

Observe that here the net liability, after five years, of the company to its policyholder is computed by taking the present value of the future benefit (\$1000 at end of year of death) less the present value of all future premiums. The reserve is thus obtained by looking towards the future, and for that reason the formula used is called *prospective*. Using the prospective principle,

reserve = value of future benefits minus value of future premiums,

we obtain the following prospective reserve formulas for four standard insurance plans:

$${}_tV_x=A_{x+t}-P_x\,a_{x+t} \qquad \text{VI}$$

$${}_tV^{\;1}_{x:\overline{n}|}=A^{\;\;1}_{x+t:\overline{n-t}|}-P^{\;1}_{x:\overline{n}|}\,a_{x+t:\overline{n-t}|} \qquad \text{VII}$$

$${}_tV_{x:\overline{n}|}=A_{x+t:\overline{n-t}|}-P_{x:\overline{n}|}\,a_{x+t:\overline{n-t}|} \qquad \text{VIII}$$

$${}_{t:m}V_x=A_{x+t}-{}_mP_x\,a_{x+t:\overline{m-t}|} \qquad (t<m) \qquad \text{IX}$$

$${}_{t:m}V_x=A_{x+t} \qquad (t\geqq m) \qquad \text{X}$$

Here the tth reserve for a \$1, n-year endowment, originally issued at age x, is obtained by considering that the policyholder after t years is aged $x+t$, and that he has endowment insurance coverage for the remaining $n-t$ years of the term of his policy; hence, the value of his future benefits is $A_{x+t:\overline{n-t}|}$. He will, however, continue to pay the net level premium, $P_{x:\overline{n}|}$, for the remaining $n-t$ years, so that the

value of his future premiums will be $P_{x:\overline{n}|}a_{x+t:\overline{n-t}|}$. In commutation symbols, this reserve formula would be

$$ {}_tV_{x:\overline{n}|}=\frac{M_{x+t}-M_{x+n}+D_{x+n}}{D_{x+t}}-P_{x:\overline{n}|}\cdot\frac{N_{x+t}-N_{x+n}}{D_{x+t}} \qquad \text{XI}$$

where

$$P_{x:\overline{n}|}=\frac{M_x-M_{x+n}+D_{x+n}}{D_x}.$$

Illustrative Example

Compute the fifth and thirtieth terminal reserves on a \$1000 twenty-payment life policy, issued at age 30. (Compare this with the illustrative example for retrospective reserve formulas, page 945.)

The premium is \$1000 ${}_{20}P_{30}=\$27.186$.

Then, from formula IX,

$$\begin{aligned}\$1000\ {}_{5:20}V_{30}&=\$1000A_{35}-\$27.186a_{35:\overline{15}|}\\&=1000(0.419883)-27.186\cdot\frac{N_{35}-N_{50}}{D_{35}}\\&=419.883-27.186(11.5552)\\&=105.74.\end{aligned}$$

From formula X,

$$\$1000\ {}_{30:20}V_{30}=\$1000A_{60}=1000(0.666718)=\$666.72.^*$$

TEST YOUR KNOWLEDGE OF PROSPECTIVE RESERVE FORMULAS

5 Justify formulas VI, VII, IX, and X, and write them in terms of commutation symbols, as was done above for formula VIII.

6 Compute by the prospective method the values of the reserves in problems 1 to 4. Check with your previous calculations.

7 Compute \$1000 ${}_{25:20}V_{30:\overline{30}|}$, the twenty-fifth reserve on a \$1000 twenty-payment, thirty-year endowment insurance to a life aged 30. (*Hint:* As the premiums have all been paid, the reserve is, prospectively, just the value of the endowment insurance benefits for the remaining five years.)

8 State circumstances under which the prospective reserve formula is simpler than the corresponding retrospective reserve formula.

EQUIVALENCE OF CORRESPONDING FORMULAS

In our examples, we have seen that the value of the reserve computed by a retrospective formula is equal to the value obtained by use of the corresponding prospective formula. One's curiosity is naturally aroused concerning the possible algebraic equivalence of the retrospective and prospective formulas. We shall now show that the formulas are algebraically identical, but, instead of giving the general proof, we shall illustrate the method of proof by considering the case of an m-payment whole life policy. The net premium for such a policy is determined by

$${}_mP_x\,a_{x:\overline{m}|}=A_x,$$

or, in commutation symbols, after multiplying by D_x,

$${}_mP_x(N_x-N_{x+m})=M_x. \qquad \text{XII}$$

* Note that this differs by one cent from the former calculation. This discrepancy is due to the fact that our commutation symbols are tabulated to only six figures, and our annual premiums to only five figures. The correct result, based on more extensive tables, is \$666.72.

This may be written as

$$ {}_mP_x(N_x - N_{x+t} + N_{x+t} - N_{x+m}) = M_x - M_{x+t} + M_{x+t} $$

and, on transposition of terms,

$$ {}_mP_x(N_x - N_{x+t}) - (M_x - M_{x+t}) = M_{x+t} - {}_mP_x(N_{x+t} - N_{x+m}). $$

Dividing by D_{x+t}, we obtain

$$ {}_mP_x \cdot {}_tu_x - {}_tk_x = A_{x+t} - {}_mP_x\, a_{x+t:\overline{m-t|}}, $$

which shows that, for $t < m$, the retrospective reserve formula is algebraically identical with the prospective formula. For $t > m$, we write XII in the form,

$$ {}_mP_x(N_x - N_{x+m}) - (M_x - M_{x+t}) = M_{x+t}, $$

and, on dividing by D_{x+t}, we again see that the retrospective formula is identical with the prospective formula.

For any policy, all that is needed in order to show that the retrospective reserve formula is identical with the corresponding prospective formula is a slight manipulation of the fundamental equation between the net level premiums and the insurance benefits.

SPECIAL FORMULAS

The reserve formulas—particularly those for endowment and whole life insurance—may be transformed in various ways to produce special formulas which may be useful if calculations are to be made on a computing machine. We illustrate some of the special forms for the case of the tth reserve of an ordinary life policy.

From formula XI, $\quad {}_tV_x = A_{x+t} - P_x\, a_{x+t}$

$\quad = P_{x+t}\, a_{x+t} - P_x\, a_{x+t},$

so that $\quad {}_tV_x = (P_{x+t} - P_x)\, a_{x+t}.$ **XIII**

In words, formula XIII states that the reserve at the end of t years must be sufficient to provide the difference in the present value of the premiums, P_{x+t}, that the life then aged $(x+t)$ would have to pay for whole life insurance from age $(x+t)$ onward, and the present value of the net level premiums, P_x, that he will actually pay from age $(x+t)$ onward.

Again, if in formula LXXXVII we substitute $1 - da_{x+t}$ for A_{x+t}, and $\frac{1}{a_x} - d$ for $P_x = \frac{A_x}{a_x} = \frac{1 - da_x}{a_x}$ (see formula LIII, page 931), we obtain

$$ {}_tV_x = 1 - da_{x+t} - \left(\frac{1}{a_x} - d\right)a_{x+t} $$

$$ = 1 - da_{x+t} - \frac{a_{x+t}}{a_x} - da_{x+t}, $$

or $\quad {}_tV_x = 1 - \frac{a_{x+t}}{a_x}.$ **XIV**

Formula XIV expresses ${}_tV_x$ in terms of annuity values only, while VI involved a single premium, an annual premium, and an annuity value.

Illustrative Example

To compute the fifth terminal reserve on a \$1000 ordinary life policy issued at age 35, we have, by formula XIII,

$$\begin{aligned}\$1000\ {}_5V_{35} &= \$1000(P_{40}-P_{35})a_{40}\\ &=(24.754-21.081)18.5598\\ &=\$68.17,\end{aligned}$$

while, by formula XIV,

$$\begin{aligned}\$1000\ {}_5V_{35} &= \$1000\left(1-\frac{a_{40}}{a_{35}}\right)\\ &=\$1000\left(1-\frac{18.5598}{19.9174}\right)\\ &=\$68.16,\end{aligned}$$

the discrepancy of one cent being here due to the fact that the premiums in Table LXXXVI are given to just five figures.

FACKLER'S ACCUMULATION FORMULA

In practice, it is necessary to have whole tables of reserves. In computing such tables, it is useful to have a method of passing from the $(t-1)$th terminal reserve to the tth reserve. Assume that we are calculating the reserves for a policy, issued at age x, providing a death benefit of \$1, for which annual premiums P are payable. We shall use the mutual fund accumulation method that was employed in developing Tables A, B, and C.

Accordingly, we suppose that a group of l_x persons at age x each took out a policy of the type in question, and at the end of $(t-1)$ years there are l_{x+t-1} survivors, for whom a total reserve fund amounting to $l_{x+t-1}\cdot {}_{t-1}V$ is held.

At the beginning of the tth year, premiums totalling $l_{x+t-1}P$ are paid, so that the total fund at the beginning of the tth year available for investment is $l_{x+t-1}({}_{t-1}V+P)$.

At the end of the tth year, d_{x+t-1} claims will be payable, and a total reserve fund of $l_{x+t}\cdot {}_tV$ will be required for the l_{x+t} survivors. We then have

$$l_{x+t-1}({}_{t-1}V+P)\,(1+i)-d_{x+t-1}=l_{x+t}\cdot {}_tV, \qquad \textbf{XV}$$

or, dividing by l_{x+t}, and rearranging factors,

$$({}_{t-1}V+P)\cdot\frac{(1+i)l_{x+t-1}}{l_{x+t}}-\frac{d_{x+t-1}}{l_{x+t}}={}_tV.$$

On multiplying both numerator and denominator by v^{x+t}, we find that

$$\frac{(1+i)l_{x+t-1}}{l_{x+t}}=\frac{D_{x+t-1}}{D_{x+t}}=\frac{N_{x+t-1}-N_{x+t}}{D_{x+t}}={}_1u_{x+t-1},$$

which is more commonly written as u_{x+t-1}; similarly

$$\frac{d_{x+t-1}}{l_{x+t}}=\frac{C_{x+t-1}}{D_{x+t}}=\frac{M_{x+t-1}-M_{x+t}}{D_{x+t}}={}_1k_{x+t-1}=k_{x+t-1}.$$

Our formula connecting ${}_{t-1}V$ and ${}_tV$ is finally written as

$$({}_{t-1}V+P)u_{x+t-1}-k_{x+t-1}={}_tV. \qquad \text{XVI}$$

To calculate a series of values of ${}_tV$, $t=1, 2, \ldots$, we should use, successively, the formulas obtained by setting $t=1, 2, \ldots$, in XVI, and observing that ${}_0V=0$, we should have

$$P \cdot u_x-k_x={}_1V$$
$$({}_1V+P)u_{x+1}-k_{x+1}={}_2V$$
$$({}_2V+P)u_{x+2}-k_{x+2}={}_3V, \text{ etc.}$$

Values of u_x and k_x, for the American Experience Table with 3% interest, are to be found in Table LXXXVI. For a policy of face amount S, formula XVI must be multiplied through by S.

TEST YOUR ABILITY TO USE FACKLER'S ACCUMULATION FORMULA

9 Use Fackler's formula to compute the first five reserves for a $1000 endowment policy issued at age 35, and check with Table B.

10 Use Fackler's formula to compute the first five reserves for a $1000 ordinary life policy issued at age 35, and check with Table C.

Initial and mean reserves

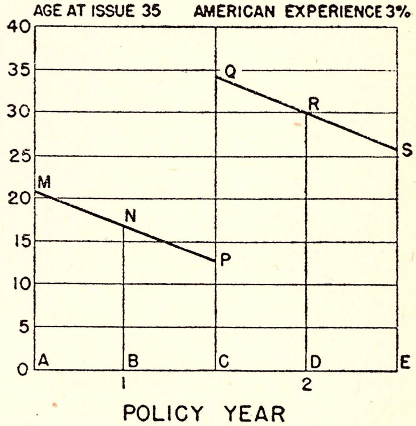
We have up to now been considering only the terminal reserve—that is, the reserve at the end of the policy year. It is sometimes necessary to consider the reserve at the beginning of the policy year after the premium then due has been paid. Such a reserve is called an *initial reserve.* If ${}_{t-1}V$ is the terminal reserve at the end of the $(t-1)$th year on a given policy, then $({}_{t-1}V+P)$ is the initial reserve at the beginning of the tth policy year.

In practice, the *mean reserve*, which is the average of the initial and terminal reserves, is important. For a policy which has in the tth policy year an initial reserve of $({}_{t-1}V+P)$ and a terminal reserve of ${}_tV$, the mean reserve for the tth policy year is $\frac{{}_{t-1}V+P+{}_tV}{2}$. The mean reserve is approximately the reserve at the middle of the policy year. (See Fig. 9.)

Each of the states has an insurance commissioner, whose duty is to supervise insurance companies which operate within the state. To the insurance commissioner, the company must submit, as of December 31 of each year, a detailed annual statement. Insurance reserves, held by the company, form one of the principal liabilities; accordingly, the company must evaluate at the end of the year the reserves on all its policies by mortality and interest standards approved by the insurance commissioner, who, in his turn, acts within the provision of the insurance law of the state. Usually the law requires that the reserves be computed on conservative mortality and interest assumptions, so that there will be assurance that the company will, with

ample margin, be able to fulfill its contracts to its policyholders. When we wish to speak specifically of these reserves required by the insurance laws, we shall refer to them as the *legal valuation reserves.*

The end of the calendar year, at which time these reserves are valued, in general will not coincide with the policy anniversaries for the various policies. However, if the policy anniversaries are distributed reasonably uniformly throughout the calendar year, then, on the average, the policy year will run from the middle of the calendar year to the middle of the next. Hence, the end of the calendar year will, on the average, be the middle of the policy year, and the reserve that should be computed is the mean reserve. The mean reserve will be less than the actual reserve at the end of the calendar year for some policies, and greater than the actual reserve for others, but, in the aggregate, will give a practical estimate of the total reserve liability as of the end of the calendar year.



Reserves for Ordinary Life Policy

AM = initial reserve in the first policy year $= 1000P_{35} = 21.08$

CP = terminal reserve in the first policy year $= 1000\,{}_1V_{35} = 12.88$

BN = mean reserve in the first policy year $= \dfrac{21.08+12.88}{2} = 16.98$

CQ = initial reserve in the second policy year $= 1000({}_1V_{35}+P_{35}) = 33.96$

ES = terminal reserve in the second policy year $= 1000\,{}_2V_{35} = 26.13$

DR = mean reserve in the second policy year $= \dfrac{33.96+26.13}{2} = 30.05.$

Fig. 9

In practice, then, the mean reserve is extensively used in computing the reserve liabilities for annual statements of insurance companies. When the year of valuation less the year of issue is $(t-1)$, the mean reserve for the tth policy year is required.

NET AMOUNT AT RISK

We return to formula XV, and on setting $l_{x+t} = l_{x+t-1} - d_{x+t-1}$, we obtain

$$l_{x+t-1}({}_{t-1}V+P)\,(1+i) - d_{x+t-1} = (l_{x+t-1} - d_{x+t-1})\,{}_tV,$$

or

$$l_{x+t-1}({}_{t-1}V+P)\,(1+i) = l_{x+t-1}\,{}_tV + d_{x+t-1}\,(1-{}_tV), \qquad \text{XVII}$$

which, on division by l_{x+t-1} becomes,

$$({}_{t-1}V+P)\,(1+i) = {}_tV + q_{x+t-1}(1-{}_tV). \qquad \text{XVIII}$$

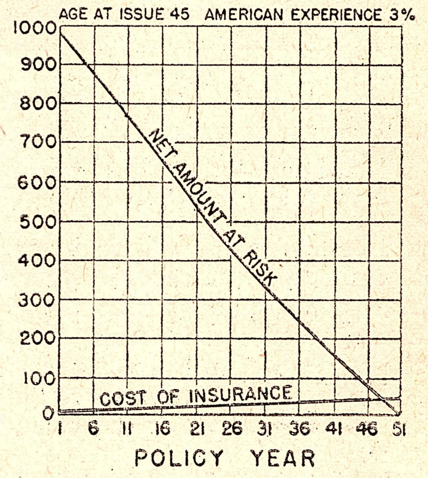
Equation XVII may be interpreted by the following reasoning: For each of the l_{x+t-1} survivors at the beginning of the tth policy year, the insurance company will have the initial reserve, $({}_{t-1}V+P)$, which during the tth year will accumulate to $({}_{t-1}V+P)\,(1+i)$, so that at

the end of the year the company will have on hand a total of $l_{x+t-1}\,({}_{t-1}V+P)\,(1+i)$. Equation XVII shows that this last sum is sufficient to provide the terminal reserve, ${}_tV$, for each of the l_{x+t-1} policyholders who began the year, and also to provide the additional amount, $1-{}_tV$, for each of the d_{x+t-1} policyholders who die. This works out exactly right, since, for each surviving policyholder, the company will have ${}_tV$, as it should, while, for each of the policyholders who dies within the year, there will be ${}_tV+1-{}_tV$, or 1, which is the amount of death benefit payable.

The additional amount, $1-{}_tV$, that is required for a policyholder who dies during the tth year, is called the *net amount at risk* in the tth year. The total additional amount required for all the death claims among the l_{x+t-1} policyholders is $d_{x+t-1}\ (1-{}_tV)$; the average additional amount per policyholder for death claims during the tth year is then

$$d_{x+t-1}\cdot\frac{1-{}_tV}{l_{x+t-1}}=q_{x+t-1}\,(1-{}_tV),$$

which is known as the *cost of insurance based upon the net amount at risk.* (For an illustration of the net amount at risk for an ordinary life policy, see Fig. 10.)



Net Amount at Risk and Cost of Insurance Based upon Net Amount at Risk $1000 Ordinary-Life; age at issue, 45; American Experience, 3%

Fig. 10

Equation XVIII shows that the initial reserve on an individual policy at the beginning of the tth year accumulates under interest to an amount sufficient to provide the terminal reserve, at the end of the tth year, and the cost of insurance based upon the net amount at risk.

Let us solve equation XVIII for P. Multiply by $v=(1+i)^{-1}$ and transpose, and there results

$$P=(v_tV-{}_{t-1}V)+vq_{x+t-1}(1-{}_tV). \qquad \textbf{XIX}$$

This equation should be of especial interest to those who want to argue about the investment features of life insurance contracts.

We consider, first, the component $(v\,{}_tV-{}_{t-1}V)$ of P. We may, if we wish, regard the reserve under a whole life or an endowment policy as a savings fund deposited with the insurance company. At the beginning of the tth year, before the premium has been paid, this savings fund will equal ${}_{t-1}V$. At the end of the tth year, the savings

fund will amount to ${}_tV$. Now notice that $v \cdot {}_tV - {}_{t-1}V$ is the *exact deposit* that must be made into the savings fund of ${}_{t-1}V$ at the beginning of the year in order that the fund may accumulate under interest to ${}_tV$ at the end of the year. For ${}_{t-1}V + (v \cdot {}_tV - {}_{t-1}V) = v{}_tV$ and this accumulates under interest in one year to $(1+i)v \cdot {}_tV = {}_tV$. Thus, the first component in formula XIX for P is the deposit required to build up the reserve considered as a savings fund.

On recalling that the one-year term premium,

$$c_x = \frac{C_x}{D_x} = \frac{v^{x+1}d_x}{v^x l_x} = v \cdot \frac{d_x}{l_x} = vq_x,$$

we see that $\qquad vq_{x+t-1} = c_{x+t-1} \qquad$ (*cf.* formula XLI, page 925).

Then the second component of P, $vq_{x+t-1}(1-{}_tV)$, is the net single premium for a one-year term insurance at age $(x+t-1)$ for an amount equal to $1-{}_tV$. This value, $1-{}_tV$, is the net amount at risk, and is also the difference between the face amount, 1, of the policy, and the amount in the savings fund, ${}_tV$, at the end of the year. Under whole life and endowment policies (see Tables B and C, pages 942 and 943), the terminal reserves steadily increase with duration, so that $1-{}_tV$ will decrease as t increases. For such policies, then, the component, $vq_{x+t-1}(1-{}_tV)$, of P, is the premium for one-year term insurance for an amount decreasing as t increases.

Gathering these two observations together, we see that the net annual premium for a whole life or endowment insurance may be considered to make the deposit sufficient to accumulate a savings fund, which at the end of each year equals the terminal reserve, and, in addition, to provide each year the decreasing term insurance required to cover the difference between the face amount of the policy in question and what will be in the savings fund at the end of the year.

TEST YOUR KNOWLEDGE OF NET AMOUNT AT RISK BY THE FOLLOWING EXERCISES

11 Using Table B, prepare for the \$1000 five-year endowment insurance at age 35 a table of premium analyses, as in Table D, page 954.

12 Using Table A, prepare for the \$1000 five-year term insurance at age 35 a table of premium analyses, as in Table D.

13 Comparing Table D with the tables prepared in exercises 11 and 12, draw conclusions about the relative importance of the savings fund and insurance components in the case of each of the three different net-level premiums.*

*As these analyses are based upon the net level premium and the level premium reserve, the important factor of insurance expenses is ignored and consequently the results correspond only in a general way with the real situation. There is another important, but rather deep-lying, difficulty in this point of view. A superficial examination of these analyses might lead one to think that a policyholder could do as well by handling the savings fund himself, merely obtaining each year sufficient term insurance to make up the difference between what would be in his fund and the amount of protection he desires, but he then would be seeking insurance on entirely different conditions than would be the case under, say, a level premium whole life contract, and the mortality rates under these new conditions can be quite different from those under the conditions of the whole life contract. In column 5 of Table D, we have assumed that the policyholder can be granted term insurance each year at net rates, based on the mortality applicable to persons who are insured under ordinary life policies. If we assume that the policyholder will maintain his own savings fund, then we must not assume, without critical examination, that he can be granted the decreasing term insurance feature at the rates applicable under ordinary life contracts.

Table D

ANALYSES OF NET PREMIUM FOR ORDINARY LIFE POLICY, ISSUED AT AGE 35

For the ordinary life policy considered in Table C, we obtain the analyses of the premium, $1000P_{35}$, into savings fund deposit and term insurance components by applying XIX for the first five policy years.

(1) Policy Year	(2) Reserve at End of Year	(3) Net Amount of Risk	(4) Savings Fund Deposit	(5) Term Insurance Component	(6) Net Annual Premium
t	${}_tV$	\$1000 − (2)	$\$1000(v\cdot{}_tV_{35}-{}_{t-1}V_{35})$	$c_{35+t-1}\times(3)$	(4) + (5)
1	\$12.88	\$987.12	\$12.505	\$8.574	\$21.08
2	26.13	973.87	12.489	8.593	21.08
3	39.76	960.24	12.472	8.609	21.08
4	53.77	946.23	12.444	8.643	21.09
5	68.16	931.84	12.405	8.673	21.08

FINAL CHECK ON YOUR KNOWLEDGE OF NET LEVEL PREMIUM RESERVES

Compute the following:

14 $1000\ {}_{10}V_{30}$ by four different methods.

15 $1000\ {}_{5:20}V_{25}$, by two methods.

16 $1000\ {}_{30:20}V_{25}$. **17** $1000\ {}_{10}V^{1}_{30:\overline{30}|}$. **18** $1000\ {}_{10}V_{30:\overline{30}|}$.

19 The mean reserves for the first two policy years on a \$1000 ordinary life policy, issued at age 25.

20 The mean reserve for the tenth policy year on a \$1000 twenty-payment life policy, issued at age 30.

21 The seventh terminal reserve on a \$2000 twenty-payment life policy, issued at age 31.

22 The sixteenth terminal reserve on a \$3000 ordinary life policy, issued at age 42.

23 The initial reserve in the sixth policy year on a \$1000 twenty-year endowment policy, issued at age 30.

MODIFIED RESERVES

In our discussion of modified reserves, we shall take a first step towards a more realistic view of insurance operations than we have obtained so far in our study of net level premiums and reserves based on net level premiums. We now take into consideration the gross premium, P', the premium actually paid by the policyholder. We let P again denote the net level premium. For simplicity of discussion, we assume that P is based on an interest rate and a mortality table reasonably close to the experience of the company. Then the net premiums, P, are sufficient to pay all death losses and set up reserves on the basis of the assumed interest and mortality rates. The excess of the gross premium over the net premium, $P'-P$, which is called the *loading*, provides for expenses and a margin for unforeseen contingencies.

In 1939, a committee of the National Association of Insurance Commissioners prepared a report on the need for a new mortality table.* In discussing net level premium reserves, the committee stated:

The principal property of the net level premium valuation method (except of course for policies with varying gross premiums) is that the annual

* *Report of the Committee to Study the Need for a New Mortality Table and Related Topics* (Actuarial Society of America, 393 Seventh Avenue, New York), p. 23.

net premium — and, therefore, the loading — are constant during the premium payment period. In other words, this valuation method assumes that the same amount is paid out or set aside for expenses each year. This was more nearly in accord with company practice at a much earlier period in the development of life insurance when expenses were distributed more evenly over the premium payment period than is now the case, but, under present practices, expenses in the first policy year are much greater than in succeeding years, due principally to the allowance of a high first-year commission rate, and a relatively low renewal rate, usually graded downward and paid for only a limited number of years. Thus, in practice, the actual expenses of operation during the first year are substantially higher than in any subsequent year. However, the net level premium method of valuation, by establishing a level loading, makes available in the first year a smaller amount than the actual expenses incurred. Hence, a financial strain which is reflected directly in the surplus is sustained.

For illustration, assume that our company charges $25.30 for ordinary life insurance for a life aged 35, and that the first-year expenses, including first-year commission, taxes, medical examination fee, amount to $20.65, while in the following years expenses are $3.00 per year.

We assume that the company maintains reserves on a 3% American Experience net level premium basis, and that its actual experience is close to these interest and mortality assumptions.

The net level premium is then $21.08, so that the loading is $4.22.

We have assumed that all the net premium is required to meet the death claims and to set up the reserves, and that only the loading is available to meet expenses.

As, in the first year, the loading is less than the expenses by $20.65 − $4.22 (= $16.43), the surplus funds of the company must be drawn upon to meet these expenses.

In the following years, the loading exceeds expenses, and eventually this excess compensates the first-year deficiency.

If our company is a young company with a small surplus, and were unfortunate enough to sell a large number of such whole life contracts, it might become technically insolvent, in that it would be unable both to pay its first-year expenses, and to maintain net level premium reserves.

On the other hand, a large, well-established organization will have surplus funds available, which will enable it to accomplish both of these objectives.

Thus, if all companies were required to maintain reserves on a net level premium basis, the young, newly-organized companies would be at a serious disadvantage, and it would be very difficult for a new company to succeed.

To continue with the Report:

The states have recognized these practical difficulties by permitting the use of suitable modifications to the net level premium reserve method. These simply require a smaller net premium for reserve purposes in the first policy year, thereby releasing a greater amount for expenses in the first year. The net premium for renewal years is increased by an amount which, over a specified number of years, will offset the first year's reduction. The present value of the increase in the net premium for renewal years is exactly equal to the reduction in the first-year net premium.

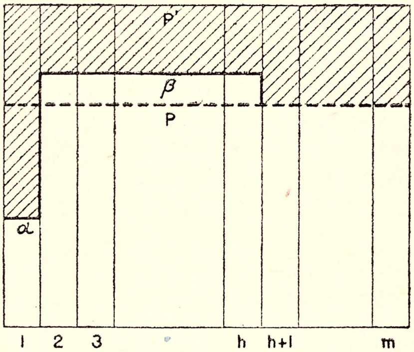
General principles

We now have in mind three systems of premiums. These are:

the gross premiums, P', P', P', . . . , payable by the policyholder;
the net level premiums, P, P, P, . . . , uniform throughout the premium payment term;
and modified net premiums of α for the first year, β for the next $(h-1)$ years, and P for the remaining years of the payment term,

$$\alpha, \beta, \beta, \ldots, \beta, P, P, \ldots,$$

where α is somewhat less than P, and β, to compensate, is somewhat greater than P. The effect upon the loading—that is, the excess of the gross premium over the net premium—is illustrated in Fig. 11, where the loading is represented by the shaded area. The use of modified net premiums permits a larger allowance—namely $P'-\alpha$—for expenses in the first year, and a smaller allowance, $P'-\beta$, in the first $(h-1)$ renewal years, than the uniform allowance, $P'-P$, under the net level premium system.



Loading in Case of Modified Net Premiums

Fig. 11

The period, h years, during which the net premiums are modified, is called the *modification period.* Reserves based upon the modified net premium will, during the modification period, be less (as shown by formula XXV) than the net level premium reserves, but, after the modification period, the net level premium is available, and, prospectively, the reserve then is just the net level premium reserve.

It is convenient to assume that the benefit is an n-year endowment insurance of the face amount of \$1, and that premiums are payable for m years, $m \leqq n$. This form includes the main standard forms except term insurance, for, if n is taken sufficiently large, the insurance benefit becomes just whole life insurance, and the policy would then be an m-payment whole life policy. By taking m also sufficiently large, the policy becomes a whole life policy with premiums payable for life—that is, an ordinary life policy. That the form does not contain the term insurance forms is no great loss, as for most term insurances the reserves are small, and it is usually considered unnecessary to arrange for modified reserves. When the number of premiums is not stated, it is to be assumed that the number is equal to the insurance term—that is, $m = n$.

It is fundamental that the present value of any system of net premiums is equal to the present value of the insurance benefits at the issue date; hence, the present value of the net level premiums is equal to the present value of the modified net premiums, as each is equal

to the present value of the benefits. For the m-payment, n-year endowment insurance, the net level and modified net premiums are:

AGE:	x	$x+1$	$x+2$	...	$x+h-1$	$x+h$	$x+h+1$	...	$x+m-1$	$x+m$
a Net level premium	P	P	P	...	P	P	P	...	P	Premiums
b Modified net premium	α	β	β	...	β	P	P	...	P	cease

$\left(\text{where fuller notation for } P \text{ is } {}_mP_{x:\overline{n}|} = \dfrac{A_{x:\overline{n}|}}{a_{x:\overline{m}|}}\right).$

Since at age x the premiums in a have present value equal to that of the premiums in b, and since, after the modification period, the premiums are the same in a and b, then the present value of the first h premiums in a equals the present value of the first h premiums in b. Then, as our fundamental relation, we obtain

$$Pa_{x:\overline{h}|} = \alpha + \beta a_{x:\overline{h-1}|}. \qquad \text{XX}$$

Replacing $a_{x:\overline{h-1}|}$ by $a_{x:\overline{h}|}-1$, we have

$$Pa_{x:\overline{h}|} = \alpha + \beta a_{x:\overline{h}|} - \beta,$$

so that

$$(\beta - P)a_{x:\overline{h}|} = \beta - \alpha, \qquad \text{XXI}$$

$$\beta = P + \frac{\beta - \alpha}{a_{x:\overline{h}|}}.$$

Hence, if P and $(\beta-\alpha)$ are known, then from XXI, β can be calculated, then α by reference to $(\beta-\alpha)$.

Illustrative Example

For example, suppose for a \$1000 twenty-payment whole life policy, issued at age 35, that 1000 $(\beta-\alpha)$ is 12.396, where 1000 α and 1000 β denote the modified net premiums, and that the modification period is the full premium term, $h=20$. Then

$$1000\,\beta = 1000\,{}_{20}P_{35} + \frac{1000\,(\beta-\alpha)}{a_{35:\overline{20}|}}$$

$$= 29.850 + \frac{12.396}{14.066}$$

$$= 29.850 + 0.881 = 30.731,$$

where ${}_{20}P_{35}$ is obtained from Table LXXXVI, and $a_{35:\overline{20}|}$ is calculated by $\dfrac{N_{35}-N_{55}}{D_{35}}$. We calculate 1000 α by the relation,

$$1000\,\alpha = 1000\,\beta - 1000\,(\beta-\alpha)$$

$$= 30.731 - 12.396 = 18.335.$$

Modified reserves, based on modified net premiums, may be calculated by using either the retrospective or the prospective principle. To develop the retrospective formula, in case $t<h$, we consider the accumulated value of the modified premium at the end of t years, and subtract the accumulated cost of insurance. The first-year premium will accumulate to

$$\alpha \cdot \frac{N_x - N_{x+1}}{D_{x+t}} = \alpha \cdot \frac{D_x}{D_{x+t}} = \frac{\alpha}{{}_tE_x}.$$

(See formulas XXV, XXVI, pages 914 and 915.)

The premiums, β, will accumulate from age $(x+1)$ for $(t-1)$ years, and hence will amount to $\beta \cdot \dfrac{N_{x+1} - N_{x+t}}{D_{x+t}} = \beta \cdot {}_{t-1}u_{x+1}.$

The accumulated cost of insurance is ${}_tk_x$, as before. We obtain, letting ${}_tV'$ denote the tth modified terminal reserve,

$$ {}_tV' = \alpha \cdot \frac{D_x}{D_{x+t}} + \beta_{t-1}u_{x+1} - {}_tk_x \text{ when } t<h \qquad \textbf{XXII} $$

(compare with formula IV).

For $t \geqq h$, it is simpler to use the prospective formulas which appear below.

To write the prospective reserve formula, we observe that, at the end of t years, the remaining benefit will be an $(n-t)$-year endowment to a life now aged $(x+t)$, of which the value is $A_{x+t:\overline{n-t}|}$. The modified renewal net premium, β, will be available for $(h-t)$ years, provided that $t<h$, and the net level premium for the remaining $(m-h)$ years. Then, using the prospective principle, that

reserve = present value of future benefits minus present value of future premiums,

we find that

$$ {}_tV' = A_{x+t:\overline{n-t}|} - \beta a_{x+t:\overline{h-t}|} - P_{h-t|}a_{x+t:\overline{m-h}|} \quad \text{when } t<h \qquad \textbf{XXIIIa} $$

$$ {}_tV' = A_{x+t:\overline{n-t}|} - Pa_{x+t:\overline{m-t}|} \quad \text{when } h \leqq t < m \qquad \textbf{XXIIIb} $$

$$ {}_tV' = A_{x+t:\overline{n-t}|} \quad \text{when } m \leqq t \leqq n \qquad \textbf{XXIIIc} $$

If $h=m$, the last term in XXIIIa is 0. We see also from XXIIIb and XXIIIc that, for $t \geqq h$, ${}_tV' = {}_tV$, the net level premium reserve.

It is interesting to compare the net level premium reserve with the modified reserve, for $t<h$. Prospectively, the net level premium reserve is

$$ {}_tV = A_{x+t:\overline{n-t}|} - Pa_{x+t:\overline{m-t}|}. $$

We may replace $a_{x+t:\overline{m-t}|}$ by $a_{x+t:\overline{h-t}|} + {}_{h-t|}a_{x+t:\overline{m-h}|}$, since an annuity for $(m-t)$ years is equivalent to an annuity for $(h-t)$ years, plus a deferred annuity, first payment at the end of $(h-t)$ years, and running for $(m-h)$ years, giving a total of $h-t+m-h(=m-t)$ payments. Then

$$ {}_tV = A_{x+t:\overline{n-t}|} - Pa_{x+t:\overline{h-t}|} - P_{h-t|}a_{x+t:\overline{m-t}|}. \qquad \textbf{XXIV} $$

Subtracting XXIIIa from XXIV, we have

$$ {}_tV - {}_tV' = (\beta - P)a_{x+t:\overline{h-t}|}. \qquad \textbf{XXV} $$

Since $\beta > P$, $(\beta - P)$ is positive; hence, ${}_tV$ exceeds ${}_tV'$ for $t<h$—that is, the net level premium reserve is greater than the modified reserve. After the modification period is passed, however, the modified reserves are equal to the net level premium reserves, as shown by XXIIIb, XXIIIc.

TEST YOUR KNOWLEDGE OF MODIFIED RESERVES BY THESE EXERCISES

24 Using the above notations, show that the additional margin for expenses permitted in the first year by the modified first-year net premium is $P-\alpha$, and that the decreased margin for renewal expenses permitted by the renewal net premium is $\beta - P$. [*Hint*: $P-\alpha = (P'-\alpha)-(P'-P)$.]

25 For a \$1000 twenty-payment whole life policy issued at age 30, $1000\,(\beta-\alpha)=10.102$. Calculate the values of $1000\,\beta$, $1000\,\alpha$, assuming $h=20$.

26 For a \$1000 thirty-year endowment insurance issued at age 30, $1000\,(\beta-\alpha)=10.102$. Calculate the values of $1000\,\beta$, $1000\,\alpha$, assuming $h=20$.

27 For a \$1000 twenty-payment, thirty-year endowment insurance issued at age 30, $1000\,(\beta-\alpha)=10.102$. Calculate the values of $1000\,\beta$, $1000\,\alpha$, assuming $h=20$.

28 Calculate by the retrospective method, and check by the prospective method (a) the first modified terminal reserve, (b) the fifth modified terminal reserve, for the policy of problem 25.

29 Calculate by the retrospective method and check by the prospective method the first modified terminal reserve for the policy of problem 27.

30 Use the prospective method to calculate the twentieth modified terminal reserve on the policy in problem 26.

31 For the policy of problem 25, calculate (a) the excess of the first net level premium terminal reserve over the first modified terminal reserve; (b) the excess of the fifth net level premium terminal reserve over the fifth modified terminal reserve. (*Hint:* Use formula XXV.)

32 Rearrange equation XX to prove the Report's statement, "The present value of the increase in the net premium for the renewal years is exactly equal to the reduction in the first-year net premium."

Full preliminary term method

This is the name applied to the modification method under which the first-year net premium is taken to be the one-year term premium at age of issue, and the modification period is the whole premium payment term, $h=m$. The one-year term premium is the lowest net premium that we should ordinarily want to hold, for any lower net premium would fail to provide for the theoretical first-year insurance costs. As the full preliminary term method employs the minimum first-year net premium, then it permits the maximum allowance for first-year expenses.

Let α_F, β_F denote the first-year and renewal net premium under the full preliminary term method. Then

$$\alpha_F=c_x=\frac{C_x}{D_x}. \qquad \text{XXVI}$$

Since the value of the modified net premiums equals the value of the benefits,

$$c_x+\beta_F a_{x:\overline{m-1|}}=A_{x:\overline{n|}},$$

or, in commutation symbols,

$$\frac{C_x}{D_x}+\beta_F\cdot\frac{N_{x+1}-N_{x+m}}{D_x}=\frac{M_x-M_{x+n}+D_{x+n}}{D_x},$$

$$\beta_F\,(N_{x+1}-N_{x+m})=(M_x-C_x)-M_{x+n}+D_{x+n}=M_{x+1}-M_{x+n}+D_{x+n}.$$

Dividing by D_{x+1}, we have

$$\beta_F a_{x+1:\overline{m-1|}}=A_{x+1:\overline{n-1|}} \qquad \text{or}$$

$$\beta_F=\frac{A_{x+1:\overline{n-1|}}}{a_{x+1:\overline{m-1|}}}={}_{m-1}P_{x+1:\overline{n-1|}}. \qquad \text{XXVII}$$

Thus, under the full preliminary term method for a policy issued at age x, the first-year net premium is the net one-year term premium, c_x, and the renewal net premium is the *net level premium* for a policy taken out at age $(x+1)$, and with premium and insurance terms one year less than the original premium and insurance terms, respectively.

For example, a twenty-payment life policy issued at age 30 would have net premiums c_{30} and ${}_{19}P_{31}$; a twenty-year endowment issued at age 30 would have net premiums c_{30} and $P_{31:\overline{19}|}$; an ordinary life policy issued at age 30 would have net premiums c_{30} and P_{31}.

The problem of reserves is then simple. As the first-year net premium provides only a one-year term insurance, the reserve at the end of the first year is 0. We may regard the renewal net premium, ${}_{m-1}P_{x+1:\overline{n-1}|}$, as the net level premium for a new policy issued at age $(x+1)$. At the end of t years from the original issue date—that is, $(t-1)$ years after attainment of age $(x+1)$, these net premiums will have accumulated the $(t-1)$th net level premium reserve on the policy issued at age $(x+1)$. In symbols, if ${}_tV'_F$ denotes the tth full preliminary term reserve,

$$\begin{aligned} {}_tV'_F &= 0 \quad \text{when } t=1 \\ {}_tV'_F &= {}_{t-1:m-1}V_{x+1:\overline{n-1}|} \quad \text{when } t>1\,. \end{aligned} \qquad \text{XXVIII}$$

For example, the tenth reserve on a twenty-payment whole life policy, issued at age 30, is

$${}_{9:19}V_{31} = A_{40} - {}_{19}P_{31}\, a_{40:\overline{10}|}$$

by formula IX; the tenth reserve on a twenty-year endowment, issued at age 30, is

$${}_9V_{31:\overline{19}|} = A_{40:\overline{10}|} - P_{31:\overline{19}|}\, a_{40:\overline{10}|}$$

by formula VIII; and the tenth reserve on an ordinary life policy, issued at age 30, is

$${}_9V_{31} = A_{40} - P_{31}\, a_{40}$$

by formula VI. These reserves may be written also in retrospective forms by use of formulas IV and V.

During the premium-payment period, the net level premium reserve, ${}_tV$, exceeds the full preliminary term reserve, ${}_tV'_F$, according to formula XXV with $h=m$, by an amount,

$${}_tV - {}_tV'_F = (\beta_F - P)a_{x+t:\overline{m-t}|}\,.$$

When all premiums are paid, the reserve under either method is equal to the value of the remaining benefit.

TEST YOUR KNOWLEDGE OF THE FULL PRELIMINARY TERM MODIFICATION

33 For each of the following policies of \$1000, issued at age 35, compute the net level premium, $1000\,P$; the full preliminary term premiums, $1000\ \alpha_F$ and $1000\ \beta_F$; the additional margin for expenses in the first year, $1000\ (P-\alpha_F)$; and the decreased margin for expenses in the renewal premiums, $1000\ (\beta_F - P)$: (a) an ordinary life policy; (b) a twenty-payment whole life policy; (c) a twenty-year endowment policy; (d) a ten-payment whole life policy; (e) a ten-year endowment policy. Check your answers in Table E.

34 For the policies in problem 33, compute the fifth terminal reserves under the full preliminary term and net level premium methods.

TABLE E

COMPARISON OF NET LEVEL AND FULL PRELIMINARY TERM EXPENSE ALLOWANCES

(*American Experience*, 3%; *age at issue*, 35; *face of policy*, $1000)

Plan	Net Level Premium	Full Preliminary Term: First-Year Net Premium	Renewal Net Premium	Additional Margin for Expense: First Year	Renewal
Ordinary life	21.081	8.686	21.737	12.395	−0.656
20-payment life	29.850	8.686	31.470	21.164	−1.620
20-year endowment	41.966	8.686	44.513	33.280	−2 547
10-payment life	49.725	8.686	55.239	41.039	−5.514
10-year endowment	89.300	8.686	100.130	80.614	−10.830

Observe that, under the full preliminary term modification for a given age of issue, the first-year net premium is the same for any plan, so that the additional margin for expense in the first year, $P-\alpha_F$, increases as P increases, and the corresponding decreases in the margin for expenses in the renewal years become substantial. It is generally considered that, for the higher premium policies, the full preliminary modification makes too much allowance for initial expenses and effects too much of a decrease in the margin for renewal expenses, but that, for policies with premiums not greater than the twenty-payment life premium, the full preliminary term method makes reasonable allowances.

Notice also that the difference between the renewal net premium and first-year net premium becomes large for the higher premium policies, increasing from 13.051 for the ordinary life policy to 91.444 for the ten-year endowment. This brings us to the idea of another modified-reserve method.

Illinois method

We shall denote the initial and renewal premiums for this modification method by α_I, β_I, I referring to the State of Illinois, in which the method was originally developed. Under this method, $h=m$ or 20, whichever is the less, and so the modification period is the full premium-payment term *only if* $m \leqq 20$. Further, for *any* policy, the difference between the renewal net premium, β_I, and the first-year net premium, α_I, is taken equal to the difference between the full preliminary term renewal premium, ${}_{19}P_{x+1}$, and the first-year net premium, c_x, for a twenty-payment whole life policy. In symbols,

$$\beta_I-\alpha_I={}_{19}P_{x+1}-c_x. \qquad \text{XXIX}$$

Under this method, for a ten-year endowment, issued at age 35,

$$1000\ (\beta_I-\alpha_I)=1000\ ({}_{19}P_{36}-c_{35})=31.470-8.686=22.784$$

(values from Table LXXXVI); but also for an ordinary life policy, issued at age 35,

$$1000\ (\beta_I-\alpha_I)=22.784,$$

where now β_I and α_I are the renewal and the first-year net premiums for an ordinary life policy. The Illinois method reduces the differences, $\beta-\alpha$, between renewal and first-year net premiums, to the same level for all plans. This has the advantage of getting rid of the large differ-

ences we observed under the full preliminary term method applied to high premium policies. On the other hand, it raises the difference for ordinary life policies, and we shall see that the method will not work for such a policy.

Formula XXIX, and the general formula, XXI, are sufficient to determine β_I and α_I completely, for, by these formulas,

$$\beta_I = P + \frac{\beta_I - \alpha_I}{a_{x:\overline{h|}}}, \text{ or}$$

$$\beta_I = P + \frac{{}_{19}P_{x+1} - c_x}{a_{x:\overline{h|}}}; \qquad \text{XXX}$$

then, from XXIX,

$$\alpha_I = \beta_I - ({}_{19}P_{x+1} - c_x).$$

Thus, for the ten-year endowment, issued at age 35, we have, using Table E for the value of 1000 $P_{35:\overline{10|}}$, Table LXXXVI for values of 1000 ${}_{19}P_{36}$ and 1000c_{35}, and $\dfrac{N_{35}-N_{45}}{D_{45}}$ for $a_{35:\overline{10|}}$,

$$1000\ \beta_I = 1000P_{35:\overline{10|}} + \frac{1000\ ({}_{19}P_{36} - c_{35})}{a_{35:\overline{10|}}}$$

$$= 89.300 + \frac{22.784}{8.4441} = 91.998,$$

and

$$1000\ \alpha_I = 1000\ \beta_I - 1000\ ({}_{19}P_{36} - c_{35})$$

$$= 91.998 - 22.784 = 69.214,$$

but, for an ordinary life policy, issued at age 35, h will be 20, so that

$$1000\ \beta = 1000P_{35} + \frac{1000\ ({}_{19}P_{36} - c_{35})}{a_{35:\overline{20|}}}$$

$$= 21.081 + \frac{22.784}{14.066} = 22.701,$$

and then

$$1000\ \alpha_I = 1000\ \beta_I - 1000\ ({}_{19}P_{36} - c_{35})$$

$$= 22.701 - 22.784 = -0.083.$$

Here α_I is negative and so certainly less than the one-year term premium, which we considered was the minimum first-year net premium that would be suitable to hold. A criterion, which will tell us whether or not the Illinois method will, for a given policy, produce a first-year net premium less than the one-year term premium will be discussed a few paragraphs later.

It is unnecessary to write down special formulas for the reserve under the Illinois method, as, once α_I and β_I are determined, the general formulas, XXII and XXIII, may be used. For example, to find the fifth terminal reserve on a \$1000 ten-year endowment, issued at age 35, we have, using the values of α_I and β_I from above, and the prospective formula, XXIIIa, with $h = m = 10$,

$$1000\ {}_5V' = 1000A_{40:\overline{5|}} - 1000\ \beta_I\, a_{40:\overline{5|}}$$

$$= \frac{1000\ (M_{40} - M_{45} + D_{45}) - 91.998\ (N_{40} - N_{45})}{D_{40}} = 439.64;$$

or, retrospectively, from formula XXII,

$$1000\ {}_5V' = \frac{1000\alpha}{{}_5E_{35}} + 1000\beta\ {}_4u_{36} - 1000\ {}_5k_{35}\,,$$

$$= 69.214 \cdot \frac{D_{35}}{D_{40}} + \frac{91.998(N_{36}-N_{40})}{D_{40}} - 1000 \cdot \frac{M_{35}-M_{40}}{D_{40}}$$

$$= \frac{69.214\ D_{35} + 91.998\ (N_{36}-N_{40}) - 1000(M_{35}-M_{40})}{D_{40}}$$

$$= 439.64\,.$$

TEST YOUR KNOWLEDGE OF THE ILLINOIS METHOD BY THESE EXERCISES

35 For each of the following policies of \$1000, issued at age 35, compute the Illinois method net premiums, 1000 β_I, 1000 α_I, and the fifth modified terminal reserve: (a) a twenty-year endowment policy; (b) a ten-payment life policy. Compare with problems 33, 34.

36 For a \$1000 thirty-year endowment policy, issued at age 35, compute the Illinois method net premiums, and the fifth and twenty-fifth modified terminal reserves. (*Hint:* $h=20$, $m=n=30$. Use the prospective method for the twenty-fifth reserve.)

Illinois standard

We return to the question of deciding whether for a given policy the Illinois method will produce a first-year net premium less than c_x. It is not very difficult, but somewhat too long for this article, to show that, if for the given policy, $\beta_F \geqq {}_{19}P_{x+1}$, then α_I is not less than c_x.*

We now have two modification methods, the full preliminary term method, which is unsuitable for high premium policies, and the Illinois method, which may be unsuitable for policies with $\beta_F < {}_{19}P_{x+1}$. There is a considerable variety in the legal standards for reserves that the various states have set up for companies doing life insurance business within the state. Since the most widely used standard originated in the State of Illinois, it is called the Illinois standard. According to this standard, policies with age of issue x are grouped into two classes:

a policies for which $\beta_F \leqq {}_{19}P_{x+1}$ — that is, the full preliminary term renewal premium for the policy is equal to, or less than the full preliminary term renewal premium for a twenty-payment whole life policy issued at age x;

b policies for which $\beta_F > {}_{19}P_{x+1}$.

For policies falling into group a, the reserve must be not less than that provided by the full preliminary term method; for policies in group b, the reserve must be not less than that provided by the Illinois modification method.

* It may also be shown that if the net level premium for the policy is greater than or equal to the twenty-payment life net premium—that is, $P \geqq {}_{20}P_x$—then α_I is not less than c_x.

TABLE F

TWENTY-FIVE-YEAR ENDOWMENT

(*American Experience*, 3%; $1000 *face amount, Illinois Standard*)

Age of Issue	$1000\ \beta_F = 1000P_{x+1:\overline{24}\rceil}$	$1000\ {}_{19}P_{x+1}$	Modification Method Permitted
41	36.72	35.79	Illinois method
42	37.18	36.63	Illinois method
43	37.70	37.53	Illinois method
44	38.28	38.47	Full preliminary term
45	38.93	39.47	Full preliminary term
46	39.66	40.53	Full preliminary term

One should be careful to distinguish the *Illinois standard* from the *Illinois modification* method. The Illinois standard is a legal requirement, which provides for the use of two modification methods, one of which is the Illinois method.

In order to compute the reserves on the policy of the company, according to the Illinois standard, each policy must be classified into group a or group b. Some policies may be classified without comparing the actual values of β_F and ${}_{19}P_{x+1}$. For instance, high premium policies, such as the ten- and twenty-year endowment, will evidently have full preliminary term renewal net premiums β_F greater than the full preliminary term renewal net premium, ${}_{19}P_{x+1}$, for a twenty-payment life policy. Hence, high premium policies will usually fall into group b. On the other hand, low premium policies, such as an ordinary life, will evidently fall in group a. This is as it should be, since the full preliminary term method, which is allowed for group a, is suitable for low premium policies, and unsuitable for those with high premiums, while the reverse is true for the Illinois modification method, which is permitted for group b.

Term policies with annual premiums would fall in group a, but, as we remarked before, these policies are usually carried with net level premium reserves.

To decide about long-term endowment policies, it may be necessary to make an actual comparison of β_F and ${}_{19}P_{x+1}$. For a $1000 twenty-five-year endowment policy, $1000\ \beta_F = 1000P_{x+1:\overline{24}\rceil} > 1000\ {}_{19}P_{x+1}$, on the American Experience 3% basis for age of issue, $x < 44$, but, from age 44 onwards, $1000\ \beta_F < 1000\ {}_{19}P_{x+1}$, as shown in Table F. Then, for $x < 44$, the Illinois standard allows the Illinois method for this policy, while for $x \geqq 44$ the Illinois standard allows the full preliminary term method.

TEST YOUR KNOWLEDGE OF THE ILLINOIS STANDARD BY THESE EXERCISES

37 State which of the following $1000 policies should be classified into group a and which into group b, on the basis of the American Experience table 3 per cent: (a) A thirty-payment whole life policy, issued at age 35; (b) A fifteen-year endowment policy, issued at age 35; (c) A thirty-year endowment policy, issued at age 27; (d) A thirty-year endowment policy, issued at age 29.

38 Compute the first and the twentieth Illinois standard terminal reserves for the policies in parts c and d of problem 37.

Other modification methods

In addition to the full preliminary term and Illinois methods, there are several other modification methods in use. The most important of these are the following:

a *Ohio method*—Let α_O and β_O denote the modified net premiums. The defining relation then is $\beta_O - \alpha_O = P_{x+1} - c_x$, and the modification period, h, equals the premium-payment period, m. By using XXI, we see that $\beta_O = P + \frac{P_{x+1} - c_x}{a_{x:\overline{m}|}}$, and then $\alpha_O = \beta_O - (P_{x+1} - c_x)$.

b *New Jersey method*—This method is applied only for $m > 20$. The modification period is taken as 20 years. The first-year net premium, $\alpha_J = c_x$, and the renewal net premium, β_J, may be obtained by solving the equation, $c_x + \beta_J a_{x:\overline{19}|} = P a_{x:\overline{20}|}$, which is equation XX applied to this case.

c *Canadian method*—We use α_C and β_C to denote the first-year and renewal net premiums. The defining relation is

$$P - \alpha_C = P_x - c_x, \text{ or } \alpha_C = P - (P_x - c_x),$$

and the modification period equals the premium payment period. Having calculated α_C, we may use equation XX to determine β_C—namely

$$\alpha_C + \beta_C a_{x:\overline{m-1}|} = P a_{x:\overline{m}|}.$$

The Ohio method, the New Jersey method, and the Canadian method appear in connection with the Ohio standard, the New Jersey standard, and the Canadian standard, respectively.*

One of the results of the study by the National Association of Insurance Commissioners of the need for a new mortality table was the development of a new valuation standard, which we shall refer to as the Commissioners' standard. For policies for level amounts of insurance, this new standard is the same as the Illinois standard, except for policies in group b and with $m > 20$. In fact, it provides some simplification of the Illinois standard.

For policies in group a, the Commissioners' standard allows the full preliminary term modification method, as does the Illinois standard. The reader will recall that, under the full preliminary term method, the modification period, h, equals the premium payment term, m.

For policies in group b, the Commissioners' standard provides that the modification period, h, shall again always be equal to m (instead of $h = m$ or 20, whichever is the less, as under the Illinois method). Let us denote by β_{Com}, the modified net renewal premiums allowed by the Commissioners' standard for policies in group b. β_{Com} is determined by

$$\beta_{Com} = P + \frac{{}_{19}P_{x+1} - c_x}{a_{x:\overline{m}|}}.$$

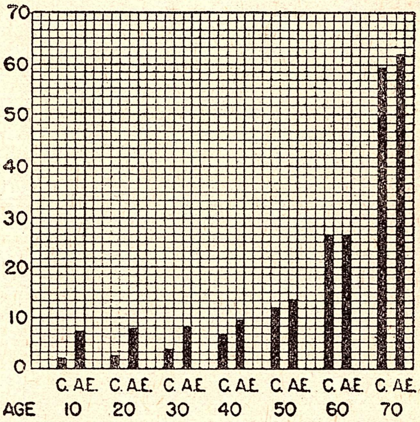
* For more details about these standards, the reader is referred to W. O. Menge's article, "Preliminary Term Valuation Methods", in Volume 25 of the *Record* of the American Institute of Actuaries.

Then β_{Com} is the same as the Illinois method renewal premium, β_I, in case $m \leqq 20$, but, for $m > 20$, $\beta_I = P + \frac{{}_{19}P_{x+1} - c_x}{a_{x:\overline{20}|}}$, and the modification period under the Illinois method is 20 instead of m.

Difficulties arise in applying the modification methods if the premiums for the policy are not uniform, or if the amount of benefit varies.

MORTALITY AND INTEREST BASES FOR RESERVES

In the past, the American Experience Table has been most widely used as the mortality basis for computing the legal valuation reserves. At present, all the states accept the valuation of ordinary policies by the American Experience Table. The American Men Table, which was based on the experience of standard ordinary insurance during the period, 1900-15, received only limited acceptance as a reserve valuation mortality table. There appeared to be some doubt that the American Men Table was safe for valuation purposes. For some plans and at some ages, it produces reserves less than the corresponding American Experience Reserves but, for the total reserve on a company's ordinary business, tends to produce a higher total than that based on the American Experience Table. Because the American Experience Table has been so extensively used as the mortality basis for the legal valuation reserve, the public has gained the impression that the American Experience Table is the basis of the majority of actuarial computations. That is not at all the true situation. Numerous and extensive studies of mortality have been made since the beginning of the century, and much of the knowledge gained from these studies has been used to distribute the cost of insurance and annuity benefits equitably among policyholders. Modern mortality experience is utilized in determining the premiums charged and, in the case of mutual companies, the distribution of surplus through annual dividends. It is now proposed that modern mortality tables shall be available as reserve valuation tables, and state laws are being amended to make this possible. (See Fig. 12.)



Deaths per Thousand according to Commissioners' and American Experience Tables

(At ages above 70, the deaths per thousand increase very rapidly and the numbers for the American Experience Table are higher than those for the Commissioners' Table.)

Fig. 12

It is intended that the Commissioners' standard may be applied by using modern mortality tables. For ordinary insurance, there is now available the Commissioners' 1941 Standard Ordinary Table (see Table

TABLE G

VARIATION OF RESERVES WITH THE RATE OF INTEREST

(*American Experience Mortality; age at issue*, 35; *face amount*, $1000)

Plan	Tenth Illinois Standard Reserve *Interest* 3%	Tenth Illinois Standard Reserve *Interest* 3½%
Ordinary life	134.86	125.48
20-payment life	242.28	219.96
20-year endowment	393.95	383.76

LXXXVII, page 989) based upon the mortality experience on policyholders with ordinary insurance during the decade, 1930-39, supplemented by additional data for the younger and older ages. By the term, *ordinary insurance*, we here mean to distinguish from industrial and group insurance. We do not mean just ordinary life policies; endowments and limited payment life policies are included, in this sense, in ordinary insurance. For industrial insurance, there has been developed the 1941 Standard Industrial Mortality Table; for annuities, the 1937 Standard Annuity Mortality Table. It is important to remark that the Commissioners' Table and the 1941 Standard Industrial Table do not show the precise experience rates. Instead, they give the experience rates plus margins, which, in the opinion of the committees preparing the tables, would offset to a reasonable extent possibly unfavorable fluctuations in the future mortality rates.

The interest rate used in the calculation of reserves is no less important than the mortality rate. A decrease in the interest rate generally causes a considerable increase in the reserve values. Rates of interest that have been required by the various states for computing the legal valuation reserves have varied from 3½% to 6%. Now some companies are valuing their reserves on the basis of 3%, or lower. To indicate how reserves vary with the interest rate, some typical American Experience reserves are shown in Table G.

The reader may obtain information about the mortality and interest bases permitted for the valuation of insurance reserves in his state of residence by consulting the insurance law of his state, or by inquiring from his State Insurance Commissioner. A summary of the bases used up to 1939 is given in the report of the committee to study the need for a new mortality table.

FINAL CHECK ON YOUR KNOWLEDGE OF MODIFIED RESERVES

Compute the following reserves, modified according to the Illinois standard:

39 Tenth terminal reserve on a $1000 ordinary life policy, issued at age 36.

40 Sixth terminal reserve on a $1000 twenty-payment whole life policy, issued at age 31.

41 Initial reserve, in the sixth policy year, for a $1000 ordinary life policy, issued at age 30. (*Hint*: Initial reserve in the sixth policy year = fifth modified terminal reserve plus the sixth modified net premium).

42 Fifth terminal reserve on a $1000 thirty-year endowment policy, issued at age 35.

43 Fourth terminal reserve on a $1000 twenty-year endowment policy, issued at age 25.

44 Mean reserve, in the sixth policy year, for the policy in problem 41.

NON-FORFEITURE BENEFITS

It happens, perhaps too often, that, for one reason or another, a policyholder will decide not to continue the payment of premiums for his insurance contract, and thus withdraws from the insured group. Opinion has crystallized into law that such a policyholder should be granted a benefit upon withdrawal, particularly if a substantial reserve is being carried for the policy. Benefits granted upon the discontinuance of premium payments by a withdrawing policyholder are called non-forfeiture benefits. A non-forfeiture benefit is either the cash value or some form of insurance that may be purchased by the cash value, applied as a single premium under suitable mortality and interest assumptions.

The most difficult problem is how the withdrawing policyholder's equity should be determined. The problem has been very fully discussed in the recent report of the Committee of the National Association of Insurance Commissioners assigned to study non-forfeiture methods and related matters.* In their summary, the committee stated:

> Equity demands that the withdrawing policyholder be allowed an amount on withdrawal, whether cash or its equivalent, which represents, with reasonable allowance for variation in views as to the assessment of expenses and other factors of operation, his contribution to the company's funds, less the cost of claims equitably assessable against his policy and less his equitable share in the expense of conducting the business, with benefit of whatever interest the company has succeeded in obtaining by the investment of these funds and less his proper contribution to stockholder's profits in a stock company or, in any case, to contingency reserves. If this amount is properly determined, continuing policyholders will be neither benefited nor harmed by his withdrawal. It is the approach to this ideal which has been sought by the committee.

Asset shares

For our present study, we shall need the concept of asset share. In the previous sections, we accumulated reserve values on the basis of net level or net modified premiums. The reserves based on net modified premiums were more realistic estimates of the funds that have accumulated, with regard to a given policy, than were the net level premium reserves. The asset share gives an even better estimate than the modified reserve. In the case of the asset share, the premium accumulated

* *Reports and Statements on Non-forfeiture Benefits and Related Matters* (Actuarial Society of America, New York), p. 149.

is the *effective premium*—that is, the gross premium which the policyholder pays less the estimated expenses for the year. (See Fig. 13.)

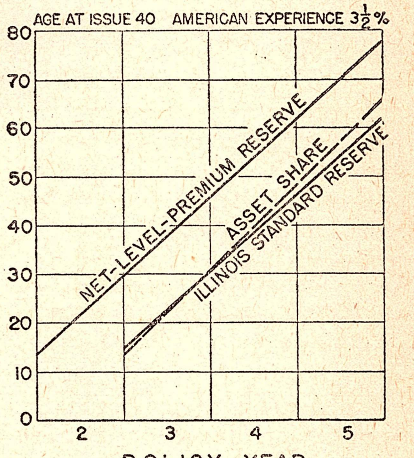
For example, let us consider the ordinary life policy, issued at age 35, with a gross premium of \$25.30, that we discussed at the beginning of the section on modified reserves. As before, we assume first-year expenses of \$20.65, renewal expenses of \$3.00 per year, interest at 3%, but we now assume that the mortality of the company is close to a modern mortality table. The effective premium in the first year is

$$\$25.30-\$20.65=\$4.65,$$

and in the renewal year is

$$\$25.30-\$3.00=\$22.30.$$

These are illustrative figures only, and may not closely represent actual practice.



Comparison of Average Asset Shares with Net Level Premium and Illinois Standard Reserves Ordinary Life Policy; age at issue, 40; American Experience, 3½%

Fig. 13

To obtain the asset shares, we should make an accumulation in much the same way as the Fackler's accumulation, discussed in the section on net level premium reserves (page 949). There would be these differences:

a The premiums accumulated would be the effective premiums, the first-year premium being much smaller than those in the renewal years.

b An adjustment would be made for the fact that claims are payable immediately upon death, rather than at the end of the year of death. A \$1000 claim, payable immediately upon death, will, on the average, be paid in the middle of the year, and is equivalent to

$$\$1000\cdot\frac{1+i}{2}=1000\ (1+0.015)=\$1015$$

at the end of the policy year.

c The values of u_x and k_x will be based on a modern mortality table close to the company's experience, which might be quite different from the mortality table used to compute the legal valuation reserves. The formulas used will be of the form,

$$1000\ B_1=1000(P_1)u_{35}-1015\ k_{35}$$
$$1000\ B_2=1000(B_1+P_2)u_{36}-1015\ k_{36}$$
$$1000\ B_3=1000(B_2+P_3)u_{37}-1015\ k_{37},$$

and so on, where 1000 B_t denotes the asset share at the end of t years, and 1000 P_t is the effective premium for the tth policy year. The resulting values will be our best estimates of the funds that will be available in respect to the given policy.

For some while, it has been thought that the values of non-forfeiture benefits should approximate the value of the asset share at the time

of withdrawal, and we shall adopt that viewpoint. Up to the present, minimum non-forfeiture benefits, required by law, have been based on the legal valuation reserve less a deduction called the surrender charge.

As the legal reserve—in the early policy years, at least—would generally be in excess of the asset share, the amount provided for non-forfeiture benefits could often be brought down to the level of the asset share by deduction of the surrender charge. This is more especially true if modified reserves are held, for then the reserves would be smaller, and would be less in excess of the asset shares.

Typical maximum-surrender charges were: for the first two policy years, the whole reserve; for the later policy years, amounts not in excess of $25 per thousand of insurance. Often, however, the surrender charges actually employed are less than these maximum values. This method of approximating the asset share by means of the legal-valuation reserve, less a surrender charge, is indirect and open to misunderstanding. Also, for some ages and plans, the method will provide values in excess of the asset share, particularly if net level premium reserves are held. Finally, the interest and mortality assumptions, which may be suitable for the valuation of policy reserves, may be less suitable for the valuation of non-forfeiture benefits.

Adjusted premiums

The committee proposed an *adjusted premium* method for determining the minimum value to be allowed a withdrawing policyholder. For an m-payment, n-year endowment policy, issued at age x, with face amount 1, the adjusted premium, P^a, is determined by a relation of the form,

$$P^a a_{x:\overline{m|}} = A_{x:\overline{n|}} + 0.02 + 0.40\ P^a + 0.25\ P^a_L \qquad \text{XXXI}$$

where P^a is the adjusted premium for the given policy, P^a_L is the adjusted premium for an ordinary life policy at the same age of issue. In words, formula XXXI states that the present value of the adjusted premiums, at the date of issue of the policy, shall equal the present value of the benefit, plus two cents per dollar of insurance, plus forty per cent of the adjusted premium, plus twenty-five per cent of the adjusted premium for an ordinary life policy at the same age.

There are some special conditions about the terms in the right member of XXXI to which the reader will do well to pay particular attention before proceeding to utilize the formula. These are:

If P^a exceeds 0.04—that is, $1000P^a$ exceeds 40—, then the term, $0.40\ P^a$, is replaced by 0.40 (0.04).

Similarly, if P^a_L exceeds 0.04, then $0.25\ P^a_L$ is replaced by 0.25 (0.04).

Further, if P^a_L exceeds P^a, then $0.25\ P^a_L$ is replaced by $0.25\ P^a$.

This sum, $0.02+0.40P^a+0.25P_L^a$, is an estimate of the excess of the first-year expenses over renewal expenses. Denoting the net level premium by

$P=\frac{A_{x:\overline{m}|}}{a_{x:\overline{m}|}}$, and $\frac{0.02+0.40\ P^a+0.25\ P_L^a}{a_{x:\overline{m}|}}$ by e, then, on dividing XXXI by $a_{x:\overline{m}|}$, we have

$$P^a=P+e, \qquad \text{XXXII}$$

which shows that P^a is the net premium plus a loading of e.

It is intended that the adjusted premiums and non-forfeiture benefits shall be calculated on the basis of the Commissioners' 1941 Standard Ordinary Mortality Table, values for which appear in Tables LXXXVII and LXXXVIII (pages 989 to 991). We observe again that this is a table based mainly on 1930-39 mortality among lives insured under standard ordinary policies, and that a margin has been added to offset possible unfavorable fluctuations. In our computations, we shall round off to six figures the values of D, C, M, N. We shall not always then get results accurate to the cent, but usually, for a \$1000 policy, the error will not be in excess of one cent.

Let us now compute the adjusted premium for a \$1000 ordinary life policy, issued at age 30, using the 3% Commissioners' Table values. From formula XXXI, we have, on multiplying through by 1000, and observing that, in this case, all adjusted premiums appearing in the formula are P_L^a,

$$1000\ P_L^a\ a_{30}=1000\ A_{30}+20+0.40\ (1000P_L^a)+0.25\ (1000\ P_L^a),$$

$$1000\ P_L^a\ (a_{30}-0.65)=100\ A_{30}+20,$$

$$1000\ P_L^a\left(\frac{N_{30}}{D_{30}}-0.65\right)=1000\cdot\frac{M_{30}}{D_{30}}+20,$$

$$1000\ P_L^a\ (N_{30}-0.65\ D_{30})=1000\ M_{30}+20\ D_{30}.$$

Then, using values, $N_{30}=8459550$, $M_{30}=134532$, $D_{30}=380927$, from Table LXXXVIII,

$$1000\ P_L^a\ (8211950)=142151000$$

$$1000\ P_L^a=17.310.$$

For the twenty-payment life policy, issued at age 30, we should have, if P^a now stands for the adjusted premium for this policy,

$$1000\ P^a\ a_{30:\overline{20}|}=1000\ A_{30}+20+0.40\ (1000\ P^a)+0.25\ (1000\ P_L^a),$$

$$1000\ P^a\ (a_{30:\overline{20}|}-0.40)=1000\ A_{30}+20+0.25\ (17.310),$$

since $1000\ P_L^a=17.310$ by the example above.

$$1000\ P^a\left(\frac{N_{30}-N_{50}}{D_{30}}-0.40\right)=1000\frac{M_{30}}{D_{30}}+24.3275,$$

$$1000\ P^a\ (N_{30}-N_{50}-0.40\ D_{30})=1000\ M_{30}+24.3275\ D_{30},$$

$$1000\ P^a\ (5442630)=143799000,$$

so that $1000\ P^a=26.421$.

For the twenty-year endowment policy issued at age 30, we have, where now P^a is the adjusted premium for this policy,

$$1000\ P^a\ a_{30:\overline{20}|}=1000\ A_{30:\overline{20}|}+20+0.40\ (1000\ P^a)+0.25\ (17.310).$$

If we went ahead with this computation, we should find that $1000\,P^a > 40$. Then we replace $0.40(1000\,P^a)$ by $0.40(40) = 16$; then, adding up the last three terms on the right,

$$1000\,P^a a_{30:\overline{20|}} = 1000\,A_{30:\overline{20|}} + 40.3275$$
$$1000\,P^a(N_{30} - N_{50}) = 1000\,(M_{30} - M_{50} + D_{50}) + 40.3275\,D_{30}$$
$$1000\,P^a(5595000) = 233328000$$
$$1000\,P^a = 41.703.$$

We shall now discuss in turn the standard non-forfeiture values, according to both the old and the new proposed methods.

Cash surrender value

This is, as the name implies, the cash amount that can be allowed a withdrawing policyholder on the discontinuance of premium payments. Upon payment of the cash surrender value, the insurance is terminated. The cash surrender value has tended to be the basic non-forfeiture value, although there is considerable opinion that the basic non-forfeiture values should be insurance rather than cash benefits. The allowance of cash values raises problems concerning the liquidity of the assets of the companies. During a depression, heavy demands for cash values at a time when assets are at a low market value may develop a serious loss for the company. It is now widely provided that payment of cash surrender values may be deferred for a period up to six months after demand therefor.

PRESENT METHOD

Under the present method, the cash surrender value is equal to the legal valuation reserve less a surrender charge and less the amount of any indebtedness against the policy. As we remarked before, the surrender charge, after the second policy year, is commonly limited to be not greater than \$25.00 per thousand of insurance. For a \$1000 ordinary life policy issued at age 30, with American Experience 3 per cent net level premium reserves, the cash value at the end of ten years is, on the basis of a surrender charge of \$10.00 per thousand,

$$\$1000\ {}_{10}V_{30} - \$10.00 = 1000\,(A_{40} - P_{30}a_{40}) - 10.00$$
$$= 459.423 - 18.283\,(18.5598) - 10.00 = \$110.09.$$

In practice, the surrender charge would vary considerably and the cash surrender value at the end of ten years might be as high as the full reserve.

For the same policy, but with Illinois standard reserves, the cash-surrender value, on the basis of a \$5.00 surrender charge, is

$$\$1000\ {}_{9}V_{31} - \$5.00 = 1000\,(A_{40} - P_{31}a_{40}) - 5.00$$
$$= 459.423 - 18.786\,(18.5598) - 5.00 = \$105.76.$$

In general, if ${}_tC$ denotes the cash-surrender value at the end of t years, for a policy with face amount of \$1, where there is no indebted-

ness against the policy, and if ${}_tV$ is the tth terminal reserve, whether net level or modified, and if ${}_tS$ is the surrender charge at the end of t years,

$$ {}_tC = {}_tV - {}_tS. \qquad \text{XXXIII} $$

The surrender charge is usually made to decrease as t increases, and, eventually, to become zero. For face amount F, the cash value is $F \cdot {}_tC$. If, further, there is a loan of $\$L$ against the policy, the cash value is $F \cdot {}_tC - L$.

PROPOSED MINIMUM CASH SURRENDER VALUES

Under the method proposed by the committee, the minimum cash surrender value, which we here denote by ${}_tC'$, will be determined by the excess of the present value of the remaining benefits over the present value of the adjusted premiums for the remainder of the premium payment term. In symbols, for an m-payment, n-year endowment, issued at age x, with face amount of \$1,

$$ {}_tC' = A_{x+t:\overline{n-t|}} - P^a a_{x+t:\overline{m-t|}}, \qquad \text{XXXIV} $$

where P^a is calculated by XXXI. This is reminiscent of the net premium reserve fomulas, and is the same except that P^a is used in place of the net premium. It should be emphasized again, however, that the mortality and interest assumptions used for evaluating ${}_tC'$ may be different from those used to calculate the legal valuation reserves. The committee recommended that the minimum non-forfeiture benefits, for ordinary insurance, be based on the Commissioners' Table.

For the \$1000 ordinary life policy issued at age 30, 1000 $P^a_L = 17.310$ (from the example above),

and
$$ \begin{aligned} 1000\ {}_{10}C' &= 1000\, A_{40} - 1000\, P^a_L\, a_{40} \\ &= 1000 \cdot \frac{M_{40}}{D_{40}} - 17.310 \cdot \frac{N_{40}}{D_{40}} \\ &= \frac{1000\ M_{40} - 17.310\ N_{40}}{D_{40}}. \end{aligned} $$

Then, using values from Table LXXXVIII, we obtain

$$ \$1000\ {}_{10}C' = \frac{307{,}095}{270{,}795} = \$113.40. $$

TEST YOUR KNOWLEDGE OF CASH SURRENDER VALUES BY THESE EXERCISES

45 For the \$1000 twenty-payment life policy issued at age 30, calculate the cash surrender value at the end of ten years on the following bases: (a) American Experience 3% net level premium reserve, surrender charge \$10.00; (b) American Experience 3% Illinois standard reserve, surrender charge \$5.00; (c) Adjusted premium method, Commissioners' Table at 3%. (See examples on adjusted premiums.)

46 For the \$1000 twenty-year endowment policy issued at age 30, calculate the cash surrender value at the end of ten years on the following bases: (a) American Experience 3% net level premium reserve, surrender charge \$10.00; (b) Adjusted premium method, Commissioners' Table at 3%.

Reduced paid-up insurance

Under this benefit, as the name implies, the withdrawing policyholder is granted an insurance for a reduced amount and for which no further payments have to be paid. The reduced insurance is on the same plan, and has the same maturity date as the original policy. The amount of insurance is determined by equating the value of the reduced benefit to the cash surrender value.

PRESENT METHOD

In general, for an m-payment, n-year endowment insurance issued at age x for the face amount of \$1, the amount of reduced paid-up insurance at the end of t years, when $t<m$, is given by solving

$$XA_{x+t:\overline{n-t}|}={}_tC \qquad \text{XXXV}$$

for X, ${}_tC$ denoting, as before, the cash-surrender value at the end of the tth policy year. (If $t\geqq m$, the policy is already paid up for the full face amount.) The mortality and interest bases used in XXXV are taken, generally, to be the same as those used for evaluating the legal reserves. For the \$1000 ordinary life policy considered above, for which the cash value, using net level premium reserves, was \$110.09, the reduced paid-up insurance benefit is given by solving

$$XA_{40}=110.09,$$

from which we find that the reduced benefit is

$$\$X=\frac{110.09}{0.459423}=\$239.63.$$

PROPOSED MINIMUM PAID-UP INSURANCE BENEFITS

The new features here are that the adjusted premium cash surrender value, ${}_tC'$, is employed, and, for ordinary insurance, the Commissioners' Table is the basis for mortality. The minimum reduced paid-up insurance benefit is given by

$$XA_{x+t:\overline{n-t}|}={}_tC',$$

where $A_{x+t:\overline{n-t}|}$ is calculated by $\dfrac{M_{x+t}-M_{x+n}+D_{x+n}}{D_{x+t}}$, with values obtainable from Table LXXXVIII for the cases which we shall consider.

For the case of the \$1000 twenty-year endowment policy, $1000\ {}_{10}C'=396.93$ (see problem 46b). Then the amount of reduced paid-up insurance at the end of ten years, is given by

$$XA_{40:\overline{10}|}=396.93,$$

$$X(M_{40}-M_{50}+D_{50})=396.93\,D_{40}.$$

Using Table LXXXVIII, we find

$$X=396.93\cdot\frac{270795}{203640}=527.83.$$

TEST YOUR KNOWLEDGE OF REDUCED PAID-UP INSURANCE BENEFITS

47 For $1000 ordinary life policy issued at age 30, calculate the reduced paid-up insurance benefit at the end of ten years which, on the Commissioners' Table with 3% interest, is equivalent to the adjusted-premium cash surrender value. (See example on page 973.)

48 For the $1000 twenty-year endowment policy issued at age 30, calculate the reduced paid-up insurance benefits at the end of ten years, which is equivalent on the basis of the American Experience 3% table to the net level premium reserve, less the surrender charge of $10.00. (See problem 46a.)

49 For the $1000 twenty-payment whole life policy issued at age 30, calculate the reduced paid-up insurance benefit at the end of ten years on the following bases: (a) The benefit is to be equivalent on the American Experience 3% basis to the net level premium reserve, less a surrender charge of $10.00; (b) The benefit is to be equivalent, on the Commissioners' Table with 3% interest, to the adjusted premium cash-surrender value. (See problem 45.)

In the early policy years, the amount of paid-up insurance available as a non-forfeiture benefit will be small for annual premium policies, and in that case a more appropriate benefit may be the extended term insurance benefit described below.

Extended term insurance

Here the insurance is continued for the full face amount, but on a term insurance basis, the length of the term being determined by comparing, on a suitable mortality and interest basis, the single premium for the term insurance with the cash-surrender value.

PRESENT METHOD

For the ordinary life policy issued at age 30, with cash value of $110.09 at the end of ten years, the length of the term on the American Experience 3% basis would be obtained by solving for t:

$$1000\ A^{1}_{40:\overline{t}|} = 110.09.$$

In general, t will not be an integral number of years. To approximate t, we write the equation in the form,

$$1000\ M_{40} - 1000\ M_{40+t} = 110.09\ D_{40},$$

whence, substituting American Experience Table 3% values for M_{40} and D_{40}, we find

$$M_{40+t} = 8364.42.$$

This shows, as we see by looking down the column of M_x values, that

$40+t$ is between 52 and 53, or t is between 12 and 13. We then calculate

$$1000\ A^{1}_{40:\overline{12|}}=1000\cdot\frac{M_{40}-M_{52}}{D_{40}}=108.09$$

$$1000\ A^{1}_{40:\overline{13|}}=1000\cdot\frac{M_{40}-M_{53}}{D_{40}}=117.19.$$

Here a difference of one year in the term is equivalent to a difference of \$117.19−\$108.09(=\$9.10) in the value. Then a difference of one day is equivalent to a difference of $\frac{\$9.10}{365}$ (=\$0.0249) in the value. The value available, \$110.09, exceeds \$108.09 by \$2.00, and this difference is equivalent to a difference of $\frac{2.00}{0.0249}$ (=80) days in the term. Hence, the cash value of \$110.09 is sufficient to provide, on the American 3% basis, a term insurance of \$1000 for 12 years and 80 days.

Let us consider the case of a \$1000 twenty-year endowment policy issued at age 30 and discontinued at age 40.

If there is no indebtedness, the cash-surrender value on the basis of the American Experience 3% net level premium reserve, surrender charge \$10.00, is \$397.51. (See problem 46a.)

This cash-surrender value exceeds \$1000 $A^{1}_{40:\overline{10|}}$=\$90.20 (on the American Experience 3% basis) which would provide term insurance for the full face amount up to the original maturity date.

One could, of course, provide term insurance for a term beyond the maturity date of the original policy, but the more usual practice is to grant term insurance for the full face amount up to the maturity date, and to apply the balance of the cash-surrender value to purchase a pure endowment, which will provide a payment if the policyholder lives to the maturity date. In our example, the amount of pure endowment would be obtained by solving for Y in the equation,

$$397.51=1000\ A^{1}_{40:\overline{10|}}+Y\,{}_{10}E_{40},$$

which equates the cash-surrender value to the value of the term insurance for the remaining ten years of the original insurance term, plus the value of the pure endowment. Here, we find, using values of D_{40} and D_{50} from Table LXXXIV, that

$$397.51=90.20+Y\cdot\frac{D_{50}}{D_{40}},$$

$$Y=462.12.$$

In general, if the cash-surrender value exceeds the value of the extended term insurance up to the original maturity date, then the amount of pure endowment that may be granted as a supplementary benefit in the case of an n-year endowment, issued at age x, and discontinued at age $(x+t)$, is given by solving for Y in the equation,

$$_{t}C=A\cdot{}^{\ 1}_{x+t:\overline{n-t|}}+Y\cdot\frac{D_{x+n}}{D_{x+t}}. \qquad \text{XXXV}$$

PROPOSED MINIMUM EXTENDED TERM INSURANCE

Under the proposed non-forfeiture law, the minimum extended term insurance benefit would be evaluated by formulas similar to those above—namely

$$ {}_tC' = A^{1}_{x+t:\overline{r|}} \qquad \text{XXXVI} $$

if r is less than the remaining insurance term, and by

$$ {}_tC' = A^{1}_{x+t:\overline{n-t|}} + Y \cdot \frac{D_{x+n}}{D_{x+t}} \qquad \text{XXXVII} $$

if r exceeds the remaining insurance term. Here ${}_tC'$ is the cash value by the adjusted premium method, and, for ordinary insurance, is to be based on the Commissioners' Table. As the mortality experienced among withdrawing policyholders, who have taken the extended term insurance benefit, has been somewhat higher than the normal rates, it is proposed that the terms on the right sides of formulas XXXVI and XXXVII may, in the case of ordinary insurances, be calculated on increased mortality rates, which, however, are not to exceed 130% of the Commissioners' Table rates.

Under former laws, companies were not required to provide non-forfeiture benefits during the first two policy years. Under the standard non-forfeiture law now proposed, an insurance non-forfeiture benefit must be provided after the policy has been in force one full year, but cash-surrender values need not be available until premiums have been paid for at least three full years. The most appropriate non-forfeiture benefit in the early policy years is generally the extended term benefit, as this does not reduce the amount of death benefit between date of discontinuance and date of possible reinstatement on a premium-payment basis.

Sometimes the amount of the extended-term insurance is taken to be the original face amount less the amount of any indebtedness that is outstanding against the policy at date of discontinuance, even though such indebtedness is discharged by a deduction from the cash-surrender value. We shall leave it to the reader to find the justification for this practice.

TEST YOUR KNOWLEDGE OF THE EXTENDED TERM INSURANCE BENEFIT

50 For the $1000 twenty-payment life policy issued at age 30, calculate the extended term insurance benefit available at the end of ten years, assuming that the benefit is equivalent on the American Experience 3% basis to the net level premium reserve less a surrender charge of $10.00.

51 For a $1000 twenty-year endowment policy issued at age 35, calculate the extended term and pure endowment benefit, if any, assuming that the benefit is equivalent, on the American Experience 3% mortality basis, to the net level premium reserve, less a surrender charge of $12.50, (a) at the end of four years; (b) at the end of fifteen years.

DIVIDENDS In determining its premium rates, a private company must, in order to meet its costs of overhead and operation, make provision for these expenses in its premium rates. There must also be some margin to provide for special contingencies, such as an epidemic, and a margin for profit.

Participating gross premiums

Participating gross premiums have been computed generally by adding to the net premium a percentage of that net premium and a percentage of the ordinary life net premium at the same age at issue—that is, by a formula of the form,

$$P' = P + eP + fP_x. \qquad \text{XXXVIII}$$

Often $e=f=\frac{1}{8}$ or 12.5%. Sometimes a constant per thousand is added to the loading. The percentages, e, f, and the constant, if used, may be graduated according to age at issue, and plan of insurance.

American Experience mortality with 3% interest has generally been used as the basis for participating premium rates, with the total loading, $P'-P$, ranging from about 20% to 30% of the net level premium.

Illustrative Example

Assuming loadings as in formula XXXVIII, with $e=f=\frac{1}{8}$, compute the gross premium for a \$1000 twenty-payment life policy, issued at age 30.

$$1000\ {}_{20}P'_{30} = 1000\left[{}_{20}P_{30} + \frac{1}{8}(P_{30} + {}_{20}P_{30})\right] = 27.186 + \frac{1}{8}(18.283 + 27.186) = 32.870.$$

TEST YOUR KNOWLEDGE OF PARTICIPATING GROSS PREMIUMS

Compute the following premiums on the American Experience 3% basis:

52 The participating gross premium for a \$1000 ordinary life policy, issued at age 31 with loading as in XXXVIII with $e=f=\frac{1}{8}$.

53 The participating gross premium for a \$1000 ordinary life policy, issued at age 30, with a loading of 25% of the net premium, plus \$2.00 per thousand.

Dividend formulas

As a result of the fact that participating gross premiums are calculated on a conservative mortality basis, and with a simple, more or less arbitrary loading, the premiums are usually more than sufficient to meet costs, and surplus funds develop. A mutual company retains a reasonable portion of the surplus as a contingency reserve and distributes the remainder to the policyholders by means of annual

dividends computed at the end of the policy years. The company expends a great deal of effort in computing equitable dividends.

One common approach to the problem is to consider that the dividend is composed of an interest profit, a mortality profit, and a loading profit. We shall follow this method, and now consider each of these factors in detail for the tth policy year for a policy of face amount $1.

a *Interest profit.* This is often taken as

$$(i'-i)I \qquad \text{XXXIX}$$

where i is the rate of interest assumed in the calculation of premiums, i' is the net rate of interest earned on investments, and $I={}_{t-1}V+P$ is the initial reserve for the tth policy year (see page 950). The theory here is that the company had the initial reserve available for investment during the policy year and has earned interest amounting to $i'I$ on this reserve, while it assumed earnings on only iI, so that a profit, $i'I-iI$, has been realized. The difficult practical problem here is the correct determination of the rate, i'.

b *Mortality profit.* If q is the mortality rate assumed and q' is the mortality rate actually experienced, then the anticipated cost of insurance based upon the net amount at risk is (see page 952)

$$C=q(1-{}_tV)$$

while the actual cost has been

$$C'=q'(1-{}_tV)$$

so that a profit,

$$C-C'=(q-q')(1-{}_tV), \qquad \text{XL}$$

has been realized. If we set $r=\frac{q'}{q}$, the ratio of the actual to expected mortality, then XL may be written as

$$C-C'=(1-r)q(1-{}_tV)=(1-r)C. \qquad \text{XLI}$$

The practical problem here is the determination of r. In practice, instead of estimating r for each individual age, we may use average values of r for groups of ages; also, r may vary according to policy plan.

c *Loading profit.* If the loading has been according to formula XXXVIII, then the actual expense may be estimated as

$$E=uP'+vP'_x+w, \qquad \text{XLII}$$

where P' is the gross premium for the policy, and P'_x is the gross premium for an ordinary life policy at the same age at issue. The coefficients, u, v, and w, would be determined so that the total of the values of E computed for all policies of a given group approximates the aggregate expenses for that group of policies. The loading profit is then

$$(L-E)\,(1+i'), \qquad \text{XLIII}$$

where L is the loading for the policy and i' is the rate of interest earned, as in a.

Putting these three profits together, we obtain as formula for the dividend, ${}_tD$, for the tth policy year:

$$ {}_tD=(i'-i)I+(1-r)C+[L-(uP'+vP'_x+w)]\,(1+i'). \qquad \textbf{XLIV} $$

Various tests of the dividends so determined would be made, including, of course, that the total of such dividends for all policies should approximate the distributable surplus.

The net cost we define as the gross premium less the annual dividend; in symbols, ${}_tNC=P'-{}_tD$. The net cost represents the effective amount that the policyholder pays for his insurance contract in the given policy year.

Illustrative Example

Here we calculate the dividend and the net cost for the fifth policy year for the \$1000 ordinary life policy, issued at age 30, on the American Experience 3% basis.

Assuming that $1000P'_{30}=1000\left(P_{30}+\frac{1}{4}P_{30}\right)$ we find that $1000P'_{30}=22.854$.

For the dividend calculation, we assume that

(a) $i'=0.035$

(b) $r=\frac{q'_{30+t-1}}{q_{30+t-1}}$ is the ratio of the mortality rate for the tth policy year by the Commissioners' Table to the rate, q_{30+t-1}, by the American Experience Table. Here $t=5$ and $r=\frac{q'_{34}}{q_{34}}=0.493$.

(c) $u=v=0.05$, and $w=0.001$—that is, \$1 per thousand. Then, to calculate $1000\,{}_5D$,

$$1000(i'-i)I=1000(0.035-0.03)\,({}_4V_{30}+P_{30})=0.31$$

$$1000(1-r)C=1000(1-0.493)q_{34}(1-{}_5V_{30})$$

$$=1000(0.507)\,(0.008831)\cdot\frac{a_{35}}{a_{30}}\ \text{(see formula XIV)}$$

$$=4.23$$

$$1000\,[L-(uP'+vP'_{30}+w)]\,(1+i')=1000\left[\frac{1}{4}P_{30}-(0.10P'_{30}+0.001)\right]1.035$$

which, since here $P'=P'_{30}$,

$$=1.34$$

Then $1000{}_5D=0.31+4.23+1.34=5.88$.

The net cost is $1000P'-1000\,{}_5D=22.85-5.88=16.97$.

TEST YOUR KNOWLEDGE OF DIVIDENDS BY THESE EXERCISES

54 Calculate, under the same assumptions, the dividend and the net cost for the tenth policy year for the ordinary life policy considered in the illustrative example. Note that now $r=\frac{q'_{39}}{q_{39}}$.

55 Calculate the interest factor in the dividend for the ninth policy year for a \$1000 twenty-payment life policy, issued at age 35, on the American Experience 3% basis, assuming $i'=0.035$.

56 Calculate the mortality factor in the dividend in the fifth policy year for a \$1000 twenty-year endowment policy, issued at age 30, on the American Experience 3% basis, assuming $r=0.600$.

Solutions to Problems and Exercises

Answers are omitted if they may be found in the illustrative tables or examples. Because the commutation columns have been cut down to six figures, discrepancies, usually not exceeding one cent per thousand of insurance, may be encountered in the calculations.

LIFE ANNUITIES AND LIFE INSURANCE NET PREMIUMS

1 \$2093.78 **2** \$823.97 **3** \$895.54
4 \$6330.92 **5** \$766.25 **6** \$10,236.24
7 \$88.38 **8** \$983.69 **9** \$182.87
10 0.0589 **11** 0.6402 **12** 0.2499
13 0.0108

14

x	l_x	d_x
20	100,000	392
21	99,608	400
22	99,208	409
23	98,799	413
24	98,386	418
25	97,968	

15 \$1760.77
16 \$9605.56 **17** \$588.09 **18** \$7998.57
19 \$405.61 **20** \$298.28 **21** \$1246.43
22 \$1292.53 **23** \$470.13 **24** \$1043.74
25 \$689.73 **26** \$919.79 **27** \$474.17
28 \$326.01 **29** \$1838.30 **30** \$123.62
31 \$959.84 **32** \$3526.52 **33** \$1288.46
34 \$353.77 **35** \$2904.38
36 (a) \$175.15 (b) \$264.90 (c) \$246.04
37 (a) \$356.18 (b) \$609.92 (c) \$909.51
38 \$2393.73 **39** \$1048.65
40 (a) \$17.81 (b) \$44.78 (c) \$21.14
41 \$4259.58 **42** \$205.24
43 \$1165.10 **44** \$11.40
45 (a) \$7.58 (b) \$8.69 (c) \$13.38 (d) \$38.96 (e) \$140.26
46 \$102.67 **47** \$3632.85 **48** \$1241.01
49 \$1446.56 **50** (a) \$7.83 (b) \$16.51
51 \$471.04 **52** \$3929.99 **53** \$63.77
54 (a) \$9 (b) \$18 **55** \$159.00
57 \$3584.05 **58** \$3523.20 **59** \$1060.09
60 \$2076.99 **61** \$6464.74 **62** \$103.57
63 \$98.18 **64** (a) \$791.26 (b) \$40.72
65 \$78.46 **66** \$9.57 (1st) **67** \$101.64,\$43.28
68 \$332.07 **69** \$50.20

LIFE INSURANCE RESERVES AND NON-FORFEITURE BENEFITS

4 (15th) \$947.98, (25th) \$1524.78
7 \$867.82 **14** \$120.10 **15** \$95.49
16 \$609.92 **17** \$31.22 **18** \$225.48
19 (1st) \$12.36, (2d) \$21.09
20 \$231.15 **21** \$313.02 **22** \$958.08
23 \$226.78
25 $1000\,\beta = 27.898$, $1000\,\alpha = 17.796$
26 $1000\,\beta = 27.460$, $1000\,\alpha = 17.358$
27 $1000\,\beta = 34.460$, $1000\,\alpha = 24.358$

28 (a) $1000\,{}_1V' = 1000\,\alpha \cdot \frac{D_{30}}{D_{31}} - 1000\,k_{30}$
(from formula XXII)
$= 17.796 \cdot u_{30} - 1000\,k_{30}$
$= 9.99 \cdot$ (u_{30}, k_{30} from Table LXXXVI.)
$1000\,{}_1V' = 1000\,A_{31} - 1000\,a_{31:19}$.
(from formula XXIIIa)
$= 392{,}089 - 27.898\ (13.696)$
$= 9.99$.

(b) $1000\,{}_5V' = 1000\,\alpha \cdot \frac{D_{30}}{D_{35}} + 1000\,\beta \cdot {}_4u_{31}$
$- 1000\,{}_5k_{30}$ (See formula XXII.)
$1000\,{}_5V' = 1000\,A_{35} - 1000\,\beta\,a_{35:15}$.
(See formula XXIIIa.)
Answer: \$97.51.

29 \$16.80 **30** \$541.47 **31** (a) \$9.75, (b) \$8.23

		Full Preliminary Term	Net Level
34	(a)	\$55.99	\$68.16
	(b)	\$98.97	\$117.52
	(c)	\$156.54	\$185.71
	(d)	\$203.87	\$229.38
	(e)	\$402.03	\$452.13

		$1000\,\beta_I$	$1000\,\alpha_I$	$1000\,{}_5V'$
35	(a)	43.586	20.802	167.16
	(b)	52.423	29.639	216.90
36		29.452	6.668	87.61

$1000\,{}_{25}V' = 745.98$

37 (a) and (d): group a; (b) and (c): group b
38 (c) \$0.53, \$540.15; (d) \$0, \$531.98
39 \$140.29 **40** \$115.68 **41** \$64.51
42 \$88.01 **43** \$129.81 **44** \$61.26
45 (a) \$220.94 (b) \$213.83 (c) \$218.94
46 (a) \$397.51 (b) \$396.93
47 \$255.46 **48** \$526.36
49 (a) \$480.91 (b) \$493.22
50 23 years, 84 days of extended term insurance
51 (a) 15 years, 103 days of extended term insurance; (b) 5 years of extended term insurance plus a pure endowment of \$743.13
52 23.483 **53** 24.854 **54** 5.28
55 1.14 **56** 2.88

Tables and Formulas

TABLE LXXXII

3% COMPOUND INTEREST

n	$(1.03)^n$	$v^n = (1.03)^{-n}$	$s_{\overline{n}\rvert}$	$a_{\overline{n}\rvert}$
1	1.030 00	0.970 874	1.0000	0.9709
2	1.060 90	0.942 596	2.0300	1.9135
3	1.092 73	0.915 142	3.0909	2.8286
4	1.125 51	0.888 487	4.1836	3.7171
5	1.159 27	0.862 609	5.3091	4.5797
6	1.194 05	0.837 484	6.4684	5.4172
7	1.229 87	0.813 092	7.6625	6.2303
8	1.266 77	0.789 409	8.8923	7.0197
9	1.304 77	0.766 417	10.1591	7.7861
10	1.343 92	0.744 094	11.4639	8.5302
11	1.384 23	0.722 421	12.8078	9.2526
12	1.425 76	0.701 380	14.1920	9.9540
13	1.468 53	0.680 951	15.6178	10.6350
14	1.512 59	0.661 118	17.0863	11.2961
15	1.557 97	0.641 862	18.5989	11.9379
16	1.604 71	0.623 167	20.1569	12.5611
17	1.652 85−	0.605 016	21.7616	13.1661
18	1.702 43	0.587 395−	23.4144	13.7535
19	1.753 51	0.570 286	25.1169	14.3238
20	1.806 11	0.553 676	26.8704	14.8775
21	1.860 29	0.537 549	28.6765	15.4150
22	1.916 10	0.521 893	30.5368	15.9369
23	1.973 59	0.506 692	32.4529	16.4436
24	2.032 79	0.491 934	34.4265	16.9355
25	2.093 78	0.477 606	36.4593	17.4131
26	2.156 59	0.463 695−	38.5530	17.8768
27	2.221 29	0.450 189	40.7096	18.3270
28	2.287 93	0.437 077	42.9309	18.7641
29	2.356 57	0.424 346	45.2189	19.1885
30	2.427 26	0.411 987	47.5754	19.6004
31	2.500 08	0.399 987	50.0027	20.0004
32	2.575 08	0.388 337	52.5028	20.3888
33	2.652 34	0.377 026	55.0778	20.7658
34	2.731 91	0.366 045−	57.7302	21.1318
35	2.813 86	0.355 383	60.4621	21.4872
36	2.898 28	0.345 032	63.2759	21.8323
37	2.985 23	0.334 983	66.1742	22.1672
38	3.074 78	0.325 226	69.1594	22.4925
39	3.167 03	0.315 754	72.2342	22.8082
40	3.262 04	0.306 557	75.4013	23.1148
41	3.359 90	0.297 628	78.6633	23.4124
42	3.460 70	0.288 959	82.0232	23.7014
43	3.564 52	0.280 543	85.4839	23.9819
44	3.671 45	0.272 372	89.0484	24.2543
45	3.781 60	0.264 439	92.7199	24.5187
46	3.895 04	0.256 737	96.5015	24.7754
47	4.011 90	0.249 259	100.3965	25.0247
48	4.132 25	0.241 999	104.4084	25.2667
49	4.256 22	0.234 950	108.5406	25.5017
50	4.383 91	0.228 107	112.7969	25.7298

TABLE LXXXIII

WHOLE LIFE ANNUITY DUE; SINGLE PREMIUM WHOLE LIFE INSURANCE

American Experience 3%

x	a_x	A_x	x	a_x	A_x
10	24.3430	.290 981	**55**	13.3928	.609 920
11	24.2247	.294 426	**56**	13.0061	.621 182
12	24.1026	.297 982	**57**	12.6172	.632 510
13	23.9765	.301 654	**58**	12.2265	.643 888
14	23.8463	.305 447	**59**	11.8348	.655 298
15	23.7118	.309 364	**60**	11.4427	.666 718
16	23.5731	.313 403	**61**	11.0509	.678 128
17	23.4298	.317 578	**62**	10.6603	.689 505
18	23.2817	.321 891	**63**	10.2716	.700 828
19	23.1289	.326 343	**64**	9.8852	.712 080
20	22.9711	.330 938	**65**	9.5022	.723 238
21	22.8083	.335 681	**66**	9.1233	.734 272
22	22.6404	.340 571	**67**	8.7495	.745 162
23	22.4672	.345 615	**68**	8.3813	.755 885
24	22.2886	.350 817	**69**	8.0198	.766 415
25	22.1044	.356 184	**70**	7.6655	.776 734
26	21.9142	.361 722	**71**	7.3192	.786 820
27	21.7182	.367 431	**72**	6.9811	.796 666
28	21.5161	.373 317	**73**	6.6509	.806 283
29	21.3077	.379 387	**74**	6.3278	.815 694
30	21.0930	.385 642	**75**	6.0108	.824 929
31	20.8716	.392 089	**76**	5.6989	.834 013
32	20.6435	.398 734	**77**	5.3915	.842 967
33	20.4084	.405 580	**78**	5.0883	.851 796
34	20.1665	.412 627	**79**	4.7897	.860 494
35	19.9174	.419 883	**80**	4.4956	.869 059
36	19.6608	.427 356	**81**	4.2085	.877 423
37	19.3969	.435 042	**82**	3.9277	.885 601
38	19.1254	.442 949	**83**	3.6521	.893 629
39	18.8465	.451 073	**84**	3.3789	.901 586
40	18.5598	.459 423	**85**	3.1069	.909 507
41	18.2655	.467 996	**86**	2.8388	.917 315
42	17.9632	.476 799	**87**	2.5793	.924 876
43	17.6531	.485 832	**88**	2.3338	.932 024
44	17.3350	.495 097	**89**	2.1029	.938 750
45	17.0093	.504 585	**90**	1.8804	.945 231
46	16.6757	.514 301	**91**	1.6625	.951 579
47	16.3348	.524 229	**92**	1.4594	.957 492
48	15.9867	.534 368	**93**	1.2939	.962 314
49	15.6318	.544 703	**94**	1.1387	.966 834
50	15.2709	.555 215	**95**	1.0000	.970 874
51	14.9045	.565 889			
52	14.5329	.576 711			
53	14.1568	.587 667			
54	13.7765	.598 743			

TABLE LXXXIV

AMERICAN EXPERIENCE TABLE OF MORTALITY

x	l_x	d_x	q_x	p_x
10	100,000	749	0.007 490	0.992 510
11	99,251	746	0.007 516	0.992 484
12	98,505	743	0.007 543	0.992 457
13	97,762	740	0.007 569	0.992 431
14	97,022	737	0.007 596	0.992 404
15	96,285	735	0.007 634	0.992 366
16	95,550	732	0.007 661	0.992 339
17	94,818	729	0.007 688	0.992 312
18	94,089	727	0.007 727	0.992 273
19	93,362	725	0.007 765	0.992 235
20	92,637	723	0.007 805	0.992 195
21	91,914	722	0.007 855	0.992 145
22	91,192	721	0.007 906	0.992 094
23	90,471	720	0.007 958	0.992 042
24	89,751	719	0.008 011	0.991 989
25	89,032	718	0.008 065	0.991 935
26	88,314	718	0.008 130	0.991 870
27	87,596	718	0.008 197	0.991 803
28	86,878	718	0.008 264	0.991 736
29	86,160	719	0.008 345	0.991 655
30	85,441	720	0.008 427	0.991 573
31	84,721	721	0.008 510	0.991 490
32	84,000	723	0.008 607	0.991 393
33	83,277	726	0.008 718	0.991 282
34	82,551	729	0.008 831	0.991 169
35	81,822	732	0.008 946	0.991 054
36	81,090	737	0.009 089	0.990 911
37	80,353	742	0.009 234	0.990 766
38	79,611	749	0.009 408	0.990 592
39	78,862	756	0.009 586	0.990 414
40	78,106	765	0.009 794	0.990 206
41	77,341	774	0.010 008	0.989 992
42	76,567	785	0.010 252	0.989 748
43	75,782	797	0.010 517	0.989 483
44	74,985	812	0.010 829	0.989 171
45	74,173	828	0.011 163	0.988 837
46	73,345	848	0.011 562	0.988 438
47	72,497	870	0.012 000	0.988 000
48	71,627	896	0.012 509	0.987 491
49	70,731	927	0.013 106	0.986 894
50	69,804	962	0.013 781	0.986 219
51	68,842	1001	0.014 541	0.985 459
52	67,841	1044	0.015 389	0.984 611
53	66,797	1091	0.016 333	0.983 667
54	65,706	1143	0.017 396	0.982 604

TABLE LXXXIV (continued)

AMERICAN EXPERIENCE TABLE OF MORTALITY

x	l_x	d_x	q_x	p_x
55	64,563	1199	0.018 571	0.981 429
56	63,364	1260	0.019 885	0.980 115
57	62,104	1325	0.021 335	0.978 665
58	60,779	1394	0.022 936	0.977 064
59	59,385	1468	0.024 720	0.975 280
60	57,917	1546	0.026 693	0.973 307
61	56,371	1628	0.028 880	0.971 120
62	54,743	1713	0.031 292	0.968 708
63	53,030	1800	0.033 943	0.966 057
64	51,230	1889	0.036 873	0.963 127
65	49,341	1980	0.040 129	0.959 871
66	47,361	2070	0.043 707	0.956 293
67	45,291	2158	0.047 647	0.952 353
68	43,133	2243	0.052 002	0.947 998
69	40,890	2321	0.056 762	0.943 238
70	38,569	2391	0.061 993	0.938 007
71	36,178	2448	0.067 665	0.932 335
72	33,730	2487	0.073 733	0.926 267
73	31,243	2505	0.080 178	0.919 822
74	28,738	2501	0.087 028	0.912 972
75	26,237	2476	0.094 371	0.905 629
76	23,761	2431	0.102 311	0.897 689
77	21,330	2369	0.111 064	0.888 936
78	18,961	2291	0.120 827	0.879 173
79	16,670	2196	0.131 734	0.868 266
80	14,474	2091	0.144 466	0.855 534
81	12,383	1964	0.158 605	0.841 395
82	10,419	1816	0.174 297	0.825 703
83	8603	1648	0.191 561	0.808 439
84	6955	1470	0.211 359	0.788 641
85	5485	1292	0.235 552	0.764 448
86	4193	1114	0.265 681	0.734 319
87	3079	933	0.303 020	0.696 980
88	2146	744	0.346 692	0.653 308
89	1402	555	0.395 863	0.604 137
90	847	385	0.454 545	0.545 455
91	462	246	0.532 468	0.467 532
92	216	137	0.634 259	0.365 741
93	79	58	0.734 177	0.265 823
94	21	18	0.857 143	0.142 857
95	3	3	1.000 000	0.000 000

TABLE LXXXV

COMMUTATION COLUMNS

American Experience 3%

x	D_x	N_x	C_x	M_x	S_x	R_x
10	74 409.4	1811 346	541.094	21651.7	36,981,455	734,216
11	71 701.0	1736 936	523.229	21110.7	35,170,109	712,564
12	69 089.4	1665 235	505.947	20587.4	33,433,173	691,454
13	66 571.2	1596 146	489.227	20081.5	31,767,938	670,866
14	64 143.0	1529 575	473.052	19592.3	30,171,792	650,785
15	61 801.7	1465 432	458.028	19119.2	28,642,217	631,192
16	59 543.6	1403 630	442.872	18661.2	27,176,785	612,073
17	57 366.4	1344 086	428.211	18218.3	25,773,155	593,412
18	55 267.4	1286 720	414.598	17790.1	24,429,069	575,194
19	53 243.0	1231 453	401.415	17375.5	23,142,349	557,404
20	51 290.9	1178 210	388.648	16974.1	21,910,896	540,028
21	49 408.3	1126 919	376.806	16585.4	20,732,687	523,054
22	47 592.4	1077 510	365.325	16208.6	19,605,768	506,469
23	45 840.9	1029 918	354.192	15843.3	18,528,257	490,260
24	44 151.5	984 077	343.398	15489.1	17,498,339	474,417
25	42 522.2	939 926	332.933	15145.7	16,514,262	458,928
26	40 950.7	897 403	323.236	14812.8	15,574,337	443,782
27	39 434.8	856 453	313.821	14489.5	14,676,933	428,969
28	37 972.4	817 018	304.681	14175.7	13,820,481	414,480
29	36 561.7	779 046	296.218	13871.0	13,003,463	400,304
30	35 200.6	742 484	287.991	13574.8	12,224,417	386,433
31	33 887.3	707 283	279.991	13286.8	11,481,933	372,858
32	32 620.3	673 396	272.590	13006.8	10,774,650	359,571
33	31 397.6	640 776	265.749	12734.2	10,101,254	346,564
34	30 217.4	609 378	259.074	12468.5	9,460,479	333,830
35	29 078.2	579 161	252.564	12209.4	8,851,100	321,362
36	27 978.7	550 082	246.882	11956.9	8,271,940	309,152
37	26 916.9	522 104	241.318	11710.0	7,721,857	297,195
38	25 891.6	495 187	236.499	11468.7	7,199,753	285,485
39	24 901.0	469 295	231.757	11232.2	6,704,567	274,017
40	23 943.9	444 394	227.685	11000.4	6,235,271	262,785
41	23 018.8	420 450	223.654	10772.7	5,790,877	251,784
42	22 124.7	397 432	220.226	10549.1	5,370,426	241,011
43	21 260.1	375 307	217.080	10328.8	4,972,995	230,462
44	20 423.8	354 047	214.724	10111.8	4,597,688	220,134
45	19 614.2	333 623	212.578	9897.03	4,243,641	210,022
46	18 830.3	314 009	211.371	9684.45	3,910,018	200,125
47	18 070.5	295 178	210.539	9473.08	3,596,009	190,440
48	17 333.6	277 108	210.515	9262.54	3,300,831	180,967
49	16 618.3	259 774	211.455	9052.03	3,023,723	171,705
50	15 922.8	243 156	213.048	8840.57	2,763,949	162,653
51	15 246.0	227 233	215.228	8627.53	2,520,793	153,812
52	14 586.7	211 987	217.935	8412.30	2,293,559	145,185
53	13 943.9	197 401	221.013	8194.36	2,081,572	136,772
54	13 316.6	183 457	224.905	7973.25	1,884,172	128,578

TABLE LXXXV (continued)

COMMUTATION COLUMNS

American Experience 3%

x	D_x	N_x	C_x	M_x	S_x	R_x
55	12 703.9	170 140	229.052	7748.34	1,700,715	120,605
56	12 104.8	157 436	233.695	7519.29	1,530,575	112,856
57	11 518.5	145 331	238.593	7285.60	1,373,139	105,337
58	10 944.5	133 813	243.706	7047.00	1,227,807	98,051.4
59	10 382.0	122 868	249.168	6803.30	1,093,995	91,004.4
60	9 830.43	112 486	254.764	6554.13	971,126	84,201.1
61	9 289.34	102 656	260.463	6299.37	858,640	77,647.0
62	8 758.32	93 366.6	266.080	6038.90	755,984	71,347.6
63	8 237.14	84 608.2	271.450	5772.82	662,618	65,308.7
64	7 725.77	76 371.1	276.575	5501.37	578,009	59,535.9
65	7 224.18	68 645.3	281.455	5224.80	501,638	54,034.5
66	6 732.31	61 421.2	285.678	4943.34	432,993	48,809.7
67	6 250.54	54 688.8	289.148	4657.67	371,572	43,866.4
68	5 779.34	48 438.3	291.784	4368.52	316,883	39,208.7
69	5 319.23	42 659.0	293.136	4076.73	268,445	34,840.2
70	4 871.16	37 339.7	293.182	3783.60	225,786	30,763.4
71	4 436.10	32 468.6	291.428	3490.42	188,446	26,979.9
72	4 015.47	28 032.5	287.447	3198.99	155,977	23,489.4
73	3 611.07	24 017.0	281.095	2911.54	127,945	20,290.4
74	3 224.79	20 405.9	272.472	2630.45	103,928	17,378.9
75	2 858.40	17 181.1	261.892	2357.97	83,521.9	14,748.5
76	2 513.25	14 322.7	249.643	2096.08	66,340.8	12,390.5
77	2 190.41	11 809.5	236.190	1846.44	52,018.0	10,294.4
78	1 890.42	9 619.09	221.761	1610.25	40,208.6	8,447.96
79	1 613.60	7 728.67	206.374	1388.49	30,589.5	6,837.71
80	1 360.22	6 115.07	190.783	1182.12	22,860.8	5,449.22
81	1 129.82	4 754.85	173.976	991.333	16,745.7	4,267.11
82	922.940	3 625.02	156.180	817.357	11,990.9	3,275.78
83	739.878	2 702.08	137.604	661.177	8,365.85	2,458.42
84	580.725	1 962.21	119.166	523.573	5,663.77	1,797.24
85	444.644	1 381.48	101.686	404.407	3,701.56	1,273.67
86	330.007	936.837	85.1229	302.721	2,320.08	869.262
87	235.272	606.829	69.2159	217.598	1,383.24	566.541
88	159.204	371.557	53.5871	148.382	776.414	348.943
89	100.980	212.353	38.8099	94.7949	404.857	200.561
90	59.2288	111.373	26.1381	55.9850	192.504	105.766
91	31.3657	52.1442	16.2148	29.8469	81.1306	49.7812
92	14.2373	20.7785	8.76715	13.6321	28.9864	19.9343
93	5.05551	6.54120	3.60354	4.86499	8.20785	6.30214
94	1.30473	1.48569	1.08577	1.26146	1.66665	1.43715
95	0.180961	0.180961	0.175690	0.175690	0.180961	0.175690

TABLE LXXXVI

ANNUAL PREMIUMS; VALUATION COLUMNS

American Experience 3%

x	$1000\ P_x$	$1000\ {}_{20}P_x$	$1000\ {}_{19}P_x$	$1000\ c_x$	u_x	K_x
20	14.407	23.131	23.944	7.577	1.038 10	.007 8660
21	14.718	23.477	24.300	7.626	1.038 15	.007 9174
22	15.043	23.833	24.668	7.676	1.038 21	.007 9694
23	15.383	24.203	25.049	7.727	1.038 26	.008 0222
24	15.740	24.585	25.443	7.778	1.038 32	.008 0757
25	16.114	24.980	25.851	7.830	1.038 37	.008 1301
26	16.506	25.391	26.274	7.893	1.038 44	.008 1967
27	16.918	25.815	26.712	7.958	1.038 51	.008 2645
28	17.351	26.256	27.165	8.024	1.038 58	.008 3333
29	17.805	26.713	27.635	8.102	1.038 67	.008 4152
30	18.283	27.186	28.122	8.181	1.038 75	.008 4985
31	18.786	27.678	28.627	8.262	1.038 84	.008 5833
32	19.315	28.189	29.153	8.356	1.038 94	.008 6819
33	19.873	28.721	29.698	8.464	1.039 06	.008 7946
34	20.461	29.274	30.265	8.574	1.039 18	.008 9096
35	21.081	29.850	30.855	8.686	1.039 30	.009 0270
36	21.736	30.452	31.470	8.824	1.039 45	.009 1720
37	22.428	31.080	32.111	8.965	1.039 60	.009 3203
38	23.160	31.736	32.781	9.134	1.039 78	.009 4976
39	23.934	32.423	33.481	9.307	1.039 97	.009 6792
40	24.754	33.143	34.213	9.509	1.040 19	.009 8913
41	25.622	33.898	34.980	9.716	1.040 41	.010 1088
42	26.543	34.693	35.787	9.954	1.040 67	.010 3587
43	27.521	35.531	36.635	10.211	1.040 95	.010 6288
44	28.561	36.416	37.529	10.513	1.041 28	.010 9474
45	29.665	37.350	38.472	10.838	1.041 63	.011 2891
46	30.841	38.341	39.470	11.225	1.042 05	.011 6970
47	32.093	39.391	40.525	11.651	1.042 51	.012 1463
48	33.426	40.506	41.645	12.145	1.043 05	.012 6677
49	34.846	41.692	42.832	12.724	1.043 68	.013 2800
50	36.358	42.954	44.093	13.380	1.044 39	.013 9740
51	37.968	44.297	45.433	14.117	1.045 20	.014 7551
52	39.683	45.730	46.860	14.941	1.046 10	.015 6294
53	41.511	47.261	48.382	15.857	1.047 10	.016 6043
54	43.461	48.900	50.008	16.889	1.048 23	.017 7036
55	45.541	50.656	51.747	18.030	1.049 49	.018 9224
56	47.761	52.541	53.612	19.306	1.050 90	.020 2885
57	50.131	54.565	55.612	20.714	1.052 45	.021 8003
58	52.663	56.742	57.761	22.268	1.054 18	.023 4739
59	55.371	59.087	60.074	24.000	1.056 11	.025 3466
60	58.266	61.616	62.565	25.916	1.058 25	.027 4254

TABLE LXXXVII

COMMISSIONERS' 1941 STANDARD ORDINARY MORTALITY TABLE

Age	Number Living l_x	Number Dying d_x	Rate of Mortality q_x	Age	Number Living l_x	Number Dying d_x	Rate of Mortality q_x
1	1,000,000	5,770	.005 77	51	800,910	10,628	.013 27
2	994,230	4,116	.004 14	52	790,282	11,301	.014 30
3	990,114	3,347	.003 38	53	778,981	12,020	.015 43
4	986,767	2,950	.002 99	54	766,961	12,770	.016 65
5	983,817	2,715	.002 76	55	754,191	13,560	.017 98
6	981,102	2,561	.002 61	56	740,631	14,390	.019 43
7	978,541	2,417	.002 47	57	726,241	15,251	.021 00
8	976,124	2,255	.002 31	58	710,990	16,147	.022 71
9	973,869	2,065	.002 12	59	694,843	17,072	.024 57
10	971,804	1,914	.001 97	60	677,771	18,022	.026 59
11	969,890	1,852	.001 91	61	659,749	18,988	.028 78
12	968,038	1,859	.001 92	62	640,761	19,979	.031 18
13	966,179	1,913	.001 98	63	620,782	20,958	.033 76
14	964,266	1,996	.002 07	64	599,824	21,942	.036 58
15	962,270	2,069	.002 15	65	577,882	22,907	.039 64
16	960,201	2,103	.002 19	66	554,975	23,842	.042 96
17	958,098	2,156	.002 25	67	531,133	24,730	.046 56
18	955,942	2,199	.002 30	68	506,403	25,553	.050 46
19	953,743	2,260	.002 37	69	480,850	26,302	.054 70
20	951,483	2,312	.002 43	70	454,548	26,955	.059 30
21	949,171	2,382	.002 51	71	427,593	27,481	.064 27
22	946,789	2,452	.002 59	72	400,112	27,872	.069 66
23	944,337	2,531	.002 68	73	372,240	28,104	.075 50
24	941,806	2,609	.002 77	74	344,136	28,154	.081 81
25	939,197	2,705	.002 88	75	315,982	28,009	.088 64
26	936,492	2,800	.002 99	76	287,973	27,651	.096 02
27	933,692	2,904	.003 11	77	260,322	27,071	.103 99
28	930,788	3,025	.003 25	78	233,251	26,262	.112 59
29	927,763	3,154	.003 40	79	206,989	25,224	.121 86
30	924,609	3,292	.003 56	80	181,765	23,966	.131 85
31	921,317	3,437	.003 73	81	157,799	22,502	.142 60
32	917,880	3,598	.003 92	82	135,297	20,857	.154 16
33	914,282	3,767	.004 12	83	114,440	19,062	.166 57
34	910,515	3,961	.004 35	84	95,378	17,157	.179 88
35	906,554	4,161	.004 59	85	78,221	15,185	.194 13
36	902,393	4,386	.004 86	86	63,036	13,198	.209 37
37	898,007	4,625	.005 15	87	49,838	11,245	.225 63
38	893,382	4,878	.005 46	88	38,593	9,378	.243 00
39	888,504	5,162	.005 81	89	29,215	7,638	.261 44
40	883,342	5,459	.006 18	90	21,577	6,063	.280 99
41	877,883	5,785	.006 59	91	15,514	4,681	.301 73
42	872,098	6,131	.007 03	92	10,833	3,506	.323 64
43	865,967	6,503	.007 51	93	7,327	2,540	.346 66
44	859,464	6,910	.008 04	94	4,787	1,776	.371 00
45	852,554	7,340	.008 61	95	3,011	1,193	.396 21
46	845,214	7,801	.009 23	96	1,818	813	.447 19
47	837,413	8,299	.009 91	97	1,005	551	.548 26
48	829,114	8,822	.010 64	98	454	329	.724 67
49	820,292	9,392	.011 45	99	125	125	1.000 00
50	810,900	9,990	.012 32				

TABLE LXXXVIII

COMMUTATION COLUMNS

Interest at 3%

Age	D_x	N_x	C_x	M_x
1	970 873.79	27 009 185.13	5438.7784	184 198.4925
2	937 157.13	26 038 311.34	3766.7231	178 759.7141
3	906 094.57	25 101 154.21	2973.7662	174 992.9910
4	876 729.70	24 195 059.64	2544.6959	172 019.2248
5	848 649.18	23 318 329.94	2273.7698	169 474.5289
6	821 657.48	22 469 680.76	2082.3274	167 200.7591
7	795 643.38	21 648 023.28	1908.0021	165 118.4317
8	770 561.30	20 852 379.90	1728.2697	163 210.4296
9	746 389.49	20 081 818.60	1536.5539	161 482.1599
10	723 113.44	19 335 429.11	1382.7143	159 945.6060
11	700 669.18	18 612 315.67	1298.9555	158 562.8917
12	678 962.38	17 911 646.49	1265.8885	157 263.9362
13	657 920.88	17 232 684.11	1264.7184	155 998.0477
14	637 493.43	16 574 763.23	1281.1565	154 733.3293
15	617 644.50	15 937 269.80	1289.3324	153 452.1728
16	598 365.52	15 319 625.30	1272.3496	152 162.8404
17	579 665.05	14 721 259.78	1266.4228	150 890.4908
18	561 515.18	14 141 594.73	1254.0590	149 624.0680
19	543 906.31	13 580 079.55	1251.3072	148 370.0090
20	526 813.06	13 036 173.24	1242.8139	147 118.7018
21	510 226.19	12 509 360.18	1243.1479	145 875.8879
22	494 122.08	11 999 133.99	1242.4082	144 632.7400
23	478 487.77	11 505 011.91	1245.0843	143 390.3318
24	463 306.15	11 026 524.14	1246.0729	142 145.2475
25	448 565.72	10 563 217.99	1254.2942	140 899.1746
26	434 246.41	10 114 652.27	1260.5294	139 644.8804
27	420 337.92	9 680 405.86	1269.2709	138 384.3510
28	406 825.79	9 260 067.94	1283.6477	137 115.0801
29	393 692.85	8 853 242.15	1299.4062	135 831.4324
30	380 926.67	8 459 549.30	1316.7577	134 532.0262
31	368 514.96	8 078 622.63	1334.7144	133 215.2685
32	356 446.79	7 710 107.67	1356.5404	131 880.5541
33	344 708.31	7 353 660.88	1378.8911	130 524.0137
34	333 289.37	7 008 952.57	1407.6736	129 145.1226
35	322 174.24	6 675 663.20	1435.6799	127 737.4490
36	311 354.85	6 353 488.96	1469.2352	126 301.7691
37	300 817.03	6 042 134.11	1504.1709	124 832.5339
38	290 551.19	5 741 317.08	1540.2458	123 328.3630
39	280 548.29	5 450 765.89	1582.4464	121 788.1172
40	270 794.53	5 170 217.60	1624.7513	120 205.6708
41	261 282.56	4 899 423.07	1671.6291	118 580.9195
42	252 000.76	4 638 140.51	1720.0088	116 909.2904
43	242 940.93	4 386 139.75	1771.2337	115 189.2816
44	234 093.74	4 143 198.82	1827.2709	113 418.0479
45	225 448.20	3 909 105.08	1884.4461	111 590.7770
46	216 997.31	3 683 656.88	1944.4676	109 706.3309
47	208 732.53	3 466 659.57	2008.3480	107 761.8633
48	200 644.59	3 257 927.04	2072.7315	105 753.5153
49	192 727.84	3 057 282.45	2142.3817	103 680.7838
50	184 972.03	2 864 554.61	2212.4172	101 538.4021

TABLE LXXXVIII (continued)

COMMUTATION COLUMNS

Interest at 3%

Age	D_x	N_x	C_x	M_x
51	177 372.08	2 679 582.58	2285.1560	99 325.9849
52	169 920.75	2 502 210.50	2359.0870	97 040.8289
53	162 612.51	2 332 289.75	2436.0957	94 681.7419
54	155 440.13	2 169 677.24	2512.7168	92 245.6462
55	148 400.03	2 014 237.11	2590.4494	89 732.9294
56	141 487.25	1 865 837.08	2668.9411	87 142.4800
57	134 697.32	1 724 349.83	2746.2451	84 473.5389
58	128 027.86	1 589 652.51	2822.9006	81 727.2938
59	121 475.98	1 461 624.65	2897.6833	78 904.3932
60	115 040.17	1 340 148.67	2969.8347	76 006.7099
61	108 719.65	1 225 108.50	3037.8848	73 036.8752
62	102 515.17	1 116 388.85	3103.3345	69 998.9904
63	96 425.956	1 013 873.680	3160.5848	66 895.6559
64	90 456.848	917 447.724	3212.5995	63 735.0711
65	84 609.582	826 990.876	3256.2023	60 522.4716
66	78 889.025	742 381.294	3290.3994	57 266.2693
67	73 300.885	663 492.269	3313.5448	53 975.8699
68	67 852.366	590 191.384	3324.0947	50 662.3251
69	62 551.988	522 339.018	3321.8732	47 338.2304
70	57 408.212	459 787.030	3305.1898	44 016.3572
71	52 430.941	402 378.818	3271.5408	40 711.1674
72	47 632.281	349 947.877	3221.4452	37 439.6266
73	43 023.492	302 315.596	3153.6502	34 218.1814
74	38 616.729	259 292.104	3067.2434	31 064.5312
75	34 424.725	220 675.375	2962.5693	27 997.2878
76	30 459.495	186 250 650	2839.5174	25 034.7185
77	26 732.807	155 791.155	2698.9868	22 195.2011
78	23 255.195	129 058.348	2542.0673	19 496.2143
79	20 035.792	105 803.153	2370.4784	16 954.1470
80	17 081.748	85 767.361	2186.6554	14 583.6686
81	14 397.565	68 685.613	1993.2818	12 397.0132
82	11 984.937	54 288.048	1793.7512	10 403.7314
83	9 842.1101	42 303.1114	1591.6280	8 609.9802
84	7 963.8179	32 461.0013	1390.8403	7 018.3522
85	6 341.0221	24 497.1834	1195.1254	5 627.5119
86	4 961.2068	18 156.1613	1008.4853	4 432.3865
87	3 808.2203	13 194.9545	834.22596	3 423.90119
88	2 863.0753	9 386.7342	675.45626	2 589.67523
89	2 104.2285	6 523.6589	534.10846	1 914.21897
90	1 508.8319	4 419.4304	411.62344	1 380.11051
91	1 053.2617	2 910.5985	308.54175	968.48707
92	714.04246	1 857.33675	224.36237	659.94532
93	468.88279	1 143.29429	157.81002	435.58295
94	297.41597	674.41150	107.12889	277.77293
95	181.62448	376.99553	69.866160	170.644041
96	106.46830	195.37105	46.225302	100.777881
97	57.141978	88.902752	30.416104	54.552579
98	25.061545	31.760774	17.632370	24.136475
99	6.6992288	6.6992288	6.5041050	6.5041050

Glossary of Life Insurance Terms

actuary: the mathematical officer of an insurance company; his work requires a knowledge of mathematics, mortality statistics, insurance law, accounting, and finance. His direct responsibility is the computation of net and gross premiums for insurances and annuities, reserves, non-forfeiture benefits, and dividends.

annuitant: the person upon whose survival the life annuity payments depend.

annuity: a sequence of regular payments, usually of equal size. The two principal types are *annuity certain* and *life annuity*.

annuity certain: an annuity of a fixed number of payments assumed certain to be made.

annuity due: an annuity in which each payment is made at the beginning of the interval associated with the payment.

annuity immediate: an annuity in which each payment is made at the end of the interval associated with the payment.

beneficiary: in life insurance, a person, named in the policy, to whom the death benefit is made payable.

deferred annuity: an annuity in which the first payment is made later than the end of one payment interval beyond the date of the contract.

dividend: the policyholder's share in the distribution of surplus.

endowment insurance policy: a life insurance contract under the terms of which the amount stated in the policy is paid to the insured if he survives a certain stated period or to the beneficiary if the insured dies during the stated period.

expectation of life: according to a given mortality table, the average length of time which a member of any given age group will live in the future.

extended insurance: a non-forfeiture benefit under which the withdrawing policyholder is granted paid-up term insurance for the original face amount, or the original face amount less the amount of any indebtedness.

gross premium: the premium paid by the policyholder to the insurance company.

life annuity: an annuity in which each payment is contingent upon the survival to the payment date of the annuitant.

loading: the excess of the gross premium over a corresponding net premium.

modified net premiums: a system of net premiums under which the first-year net premium is taken to be smaller than the following net premiums.

modified reserve: a reserve based on the accumulation of modified net premiums.

mortality table: a table showing rates of mortality, and derived functions.

net premium: a complete or partial payment for a life insurance policy computed without provision for expenses of any kind nor for profit to the insurer.

non-forfeiture benefit: a benefit available on discontinuance of premium payments by a withdrawing policyholder.

non-participating insurance: insurance with minimum premium rates, which does not entitle the policyholder to share in the distribution of surplus.

ordinary life policy: a whole life policy paid for by means of level annual premiums paid throughout the entire future life of the insured.

policy: a life insurance contract.

premium: a payment made by the insured for his insurance policy.

renewal premium: a premium for a policy year after the first.

reserve: the excess of accumulated premiums over accumulated cost, or the present value of future benefits, less the present value of future premiums.

surplus: excess of assets over liabilities.

surrender: discontinuance of the original policy.

term insurance policy: a life insurance contract under which the death benefit is payable only if the insured dies within a stated interval of time.

valuation: the calculation of the reserves on all the company's policies in accordance with the mortality and interest bases required by law.

STATEMENT OF THE OWNERSHIP, MANAGEMENT, CIRCULATION, ETC., REQUIRED BY THE ACTS OF CONGRESS OF AUGUST 24, 1912, AND MARCH 3, 1933, of PRACTICAL MATHEMATICS, published quarterly, at Dunellen, N. J., for October 1, 1943.

State of New York }
County of New York } ss:

Before me, a Notary Public in and for the State and county aforesaid, personally appeared A. R. Mahony, who, having been duly sworn according to law, deposes and says that he is the Business Manager of the PRACTICAL MATHEMATICS, and that the following is, to the best of his knowledge and belief, a true statement of the ownership, management, etc., of the aforesaid publication for the date shown in the above caption, required by the Act of August 24, 1912, as amended by the Act of March 3, 1933, embodied in section 537, Postal Laws and Regulations, to wit:

1. That the names and addresses of the publisher, editor, managing editor, and business managers are: Publisher, John J. Crawley, 23 Shawnee Rd., Scarsdale, N. Y.; Editor, Reginald S. Kimball, 37 West 47th Street, New York, N. Y.; Managing Editor, Frank W. Price, 37 West 47th Street, New York, N. Y.; Business Managers, A. R. Mahony, Midland Gardens, Bronxville, N. Y.

2. That the owner is: National Educational Alliance, Inc., 37 West 47th St., New York, N. Y.; Wm. H. Wise & Co., Inc., 50 West 47th St., New York, N. Y.

3. That the known bondholders, mortgagees, and other security holders owning or holding 1 per cent or more of total amount of bonds, mortgages, or other securities are: None.

4. That the two paragraphs next above, giving the names of the owners, stockholders, and security holders, if any, contain not only the list of stockholders and security holders as they appear upon the books of the company but also, in cases where the stockholder or security holder appears upon the books of the company as trustee or in any other fiduciary relation, the name of the person or corporation for whom such trustee is acting, is given; also that the said two paragraphs contain statements embracing affiant's full knowledge and belief as to the circumstances and conditions under which stockholders and security holders who do not appear upon the books of the company as trustees, hold stock and securities in a capacity other than that of a bona fide owner; and this affiant has no reason to believe that any other person, association, or corporation has any interest direct or indirect in the said stock, bonds, or other securities than as so stated by him.

A. R. MAHONY, Business Manager.

Sworn to and subscribed before me this 28 day of September 1943.

(SEAL) Albert B. Beeland
Notary Public, New York City
(My commission expires March 30, 1945)